Lecture 6

Thursday, October 15, 2020 6:42 AM

Baker-Akhiezer functions II

Recall: Let (Γ, a, b) be a smooth genus g algebraic curve with a chosen basis of cycles $a_i, b_i \in H_1(\Gamma, \mathbb{Z})$ $i=1,\dots,g$ with the canonical matrix of intersections

$$a_i \circ a_j = b_i \circ b_j = 0 \qquad a_i \circ b_j = \delta_{ij}$$

Then we consequently defined:

- The basis of normalized holomorphic differentials ω_k, $k=1,\dots,g$

$$\oint_{a_i} \omega_j = \delta_{ij} \qquad \checkmark$$

- The matrix of b-periods

$$B_{ij} = \oint_{b_i} \omega_j = B_{ji}, \quad \operatorname{Im} B > 0 \qquad \checkmark$$

- The Jacobian

$$J(\Gamma) = \mathbb{C}^g / \mathbb{I} n + Bm \qquad \checkmark$$

$$n = (n_1, \dots, n_g)^t \qquad m = (m_1, \dots, m_g)^t \in \mathbb{Z}^g$$

- The Abel map $A: \underline{\Gamma} \to \mathbb{C}^g \to \underline{J(\Gamma)}$

$$\underline{A_k(p)} = \int_{q_0}^{p} \underline{\omega_k}$$

- The Riemann's theta function

$$\theta(z) := \theta(z|B) = \sum_{m \in \mathbb{Z}^g} e^{2\pi i (z, m) + \pi i (m^t, Bm)}$$

$$\left| \begin{array}{l} \theta(z + e_i) = \theta(z) \qquad\qquad z = (z_1, \dots, z_g) \\ \theta(z + B_k) = e^{-2\pi i z_k - \pi i B_{kk}} \theta(z) \end{array} \right.$$

- Jacobi inversion theorem

For a generic $Z \in J(\Gamma)$ the function

$$\underline{\theta(A(p) + Z)}$$

has g zeros $\gamma_1, \dots, \gamma_g$

$$\underline{Z = -\sum_{s=1}^{g} A(\gamma_s) + \mathcal{K}} \quad \leftarrow \text{Riemann const}$$

Corr Let $Z_0 = -\sum_{s=1}^{g-1} A(\gamma_s) + \mathcal{K}$

Then $\boxed{\theta(A(p) - A(q) + Z_0)}$ has zeros at $q, \gamma_1, \dots, \gamma_{g-1}$

$\Rightarrow$ a)

$$d\Omega_{q,q_0}(p) = d_p \ln \theta(A(p) - A(q) + Z_0) - - d_p(\ln \theta\, A(p) - A(q_0) + Z) \qquad \frac{dz}{z-q} - \frac{dz}{z - q_0}$$

is a unique normalized differential with simple poles (res ± 1) at q, q_0

$$\oint_{a_i} d\Omega_{q,q_0} = 0$$

b) Let z be a coordinate near q_0, $q = q(z)$

Then for $k > 1$

$$\boxed{d\Omega_k := \frac{1}{k!} \frac{\partial^k}{\partial z^k} d_p \ln \theta(A(p) - A(q(z)) + Z_0)}$$

is the meromorphic differential with the pole at q_0 of the form

$$\approx d(z^{-k}) + O(1)$$

s.t.

$$\underline{\oint_{a_i} d\Omega_k = 0}$$

c) Functions on Γ:

Given $\boxed{\gamma_1, \dots, \gamma_{g-1}, \gamma_g, \dots \gamma_{g+r-1}}$ $\deg \mathcal{D}_i = g+r-1-$

$P_1, \dots, P_r$ $r-1=g$

$$\tilde{r}_i(p) = \frac{\left[\prod_{j\neq i} \theta(A(p)-A(P_j)+z_0)\right]\theta(A(p)-A(P_i)+\hat{z})}{\prod_{s=g}^{g+r-1}\theta(A(p)-A(\gamma_s)+z_0)}$$

$$z_0 = -\sum_{s=1}^{g-1} A(\gamma_s) + K$$

$$\hat{z} = -\sum_{s=1}^{g+r-1} A(\gamma_s) + \sum A(P_i) + K$$

is the function with poles at $\gamma_1 \dots \gamma_{g+r-1}$ and zeros at $P_1, \dots, \hat{P}_i, \dots P_r$

One can uniquely normalize it

$$\boxed{r_i(p)} = \frac{\tilde{r}_i(p)}{r_i(P_i)}$$

Th Γ, P_α, z_α local coordinate at P_α

$$\psi(t,p) = \exp\left(\sum_i t_{\alpha_i}\Omega_{\alpha,i}(p)\right)\frac{\theta(A(p)+\sum_{\alpha i} t_{\alpha i}U_{\alpha i}+z)}{\theta(A(p)+z)}$$

where

$d\Omega_{\alpha i}$ is the normalized meromorphic differential with pole at P_α of the form $d(z_\alpha^{-i}+O(1))$

$U_{\alpha i}$ is the vector with coordinates

$$U^k_{\alpha,i} = \frac{1}{2\pi\sqrt{-1}}\oint_{b_k} d\Omega_{\alpha,i}$$

is the BA function with the divisor of poles at $\gamma_1 \dots, \gamma_g$

$$z = -\sum_{s=1}^{g} A(\gamma_s) + K$$

Given $\gamma_1, \dots \gamma_{g+r-1}$, $P_1 \dots P_r$

$\gamma_1 \dots \gamma_{g+r-1}$

$$\psi_i(t,p) = r_i(p)\exp\left(\sum_{\alpha,j} t_{\alpha j}\Omega_{\alpha,j}(p) - \Omega_{\alpha,j}(p_i)\right)$$

$$\cdot\frac{\theta(A(p)-A(p_i)+\sum t_{\alpha j}U_i+\hat{z})\ \theta(\hat{z})}{\theta(A(p)-A(p_i)+\hat{z})\ \theta(\sum t_{\alpha j}U_j+\hat{z})}$$

$(r_i)_\infty = \gamma_1, \dots \gamma_{g+r-1}$ $\quad r_{i\,0} = P_1 + \dots + P_{i-1} + P_{i+1} + P_r$

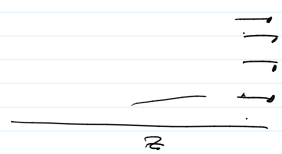
Riemann Roc...

Rigid logy

$$\dot{L} = [A, L]$$

$$L = \mathcal{J}^2 z + \Omega \qquad , A = \mathcal{J} z + V$$

$$\rightarrow \quad \det\left(k\mathbb{1} - \mathcal{J}^2 z + \Omega\right) = 0$$



$$\xi_1^{ij} = \frac{\theta\left(A(p_j) - A(p_i) + \sum t_{\alpha,k} U_k + \hat{Z}\right)}{\theta\left(\sum t_{\alpha,k} U_k + \hat{Z}\right)}$$

const independent on t

$$\Omega = \left[\mathcal{J}^2, \quad \xi_1^{ij}\right]$$

$$\underset{\sim}{L} \hat{\Psi} = \hat{\Psi} \hat{k}$$

$$\hat{\Psi} = \left(1 + \sum \xi_s z^{-s}\right) e^{f}$$

$$L = \mathcal{J}^2 z + \Omega \qquad \hat{k} = \mathcal{J}^2 z + O(1)$$

$$\left(\mathcal{J}^2 z + \Omega\right)\left(1 + \sum \xi_r z^{-s}\right) = \left(1 + \sum \xi_s z^{-s}\right)\left(\mathcal{J}^2 z + O(1)\right)$$

$$\Omega = \xi_1 \mathcal{J}^2 - \mathcal{J}^2 \xi_1$$