Lecture 5

Thursday, October 8, 2020 9:59 AM

Baker-Akhiezer function

Recall

Let Γ be a smooth genus g algebraic curve with fixed local coordinates z_α near marked points P_α

$t = (t_{\alpha,i})$ $i = 0, \dots, \infty$

$t_{\alpha,0} \in \mathbb{Z}$ $\sum_\alpha t_{\alpha,0} = 0$

Thm For a generic divisor $\mathcal{D}$ of degree $d \geq g$ the space $\mathcal{L}(t,\mathcal{D})$ of function $\psi(t,p)$ $p \in \Gamma$ s.t.

a) $\psi(t,p)$ is meromorphic on $\Gamma \setminus \{P_\alpha\}$

$(\psi) + \mathcal{D} \geq 0$

BA functions b) near P_α

$$\psi(t, z_\alpha) = z_\alpha^{t_{\alpha,0}} e^{\sum_{i=1}^{\infty} t_{\alpha,i} z_\alpha^{-i}} \times \qquad z_\alpha = z_\alpha(p)$$

$$\times \left(\sum_{s=0}^{\infty} \xi_{s,\alpha}(t) z_\alpha^s \right)$$

is dimension $d - g + 1$

Proof By explicite construction

Γ algebraic curve genus g

On any topological real two-dimension surface introduce a metric and coordinates

$$ds^2 = e^\varphi (dx^2 + dy^2)$$

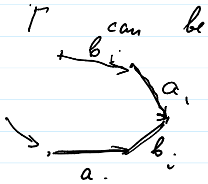
$$z = x + iy \qquad x', y'$$

$$z' = z'(z)$$

$$g = \dim (\text{holomorphic one-forms})$$

$$\mathbb{C}^{2g} = H^1(\Gamma, \mathbb{C}) = H^{1,0}(\Gamma, \mathbb{C}) + H^{0,1}(\Gamma, \mathbb{C})$$

Γ can be presented as



$i = 1, \dots, g$

$a_i, b_i \in H_1(\Gamma, \mathbb{Z})$

$a_i \circ a_j = b_i \circ b_j = 0$

$a_i \cdot b_j = \delta_{ij}$

$$a_i \circ a_j = b_i \circ b_j = 0$$

$$a_i \cdot b_j = \delta_{ij}.$$

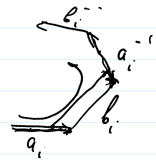
Let ω_i be a basis of normalized holomorphic diff

$$\oint_{a_j} \omega_i = \delta_{ij}$$

Bilinear Riemann identities

Consider

$$0 < \int_\Gamma \omega \wedge \bar\omega = \oint_{\partial\Gamma} v\,\bar\omega =$$



$$dv = \omega$$

$$\int_{a_i} v\bar\omega - \int_{a_i'} v\bar\omega = B_i \bar A_i$$

$$\int_{b_i} v\bar\omega - \int_{b_i'} v\bar\omega = -A_i \bar B_i$$

$$0 < \omega\wedge\bar\omega = -\sum_{i=1}^{g}\left(A_i \bar B_i - \bar A_i B_i\right) \Rightarrow A_i \neq 0$$

<u>Introduce</u> b-periods

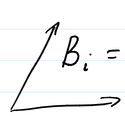
- $$B_{ij} = \oint_{b_i} \omega_j = \int_{b_j} \omega_i = B_{ji}$$

$$0 = \int \omega_i \wedge \omega_j = \oint v_i\,\omega_j$$

- $\operatorname{Im} B > 0$

<u>Jacobian of</u> Γ $\quad J(\Gamma) = \mathbb{C}^g / n + Bm = T^g$

$$n \in \mathbb{Z}^g \quad m \in \mathbb{Z}^g$$



$B_i = (B_{ij})$

e_i

Riemann theta-function

$$\theta(z\,|\,B) = \sum_{m\in\mathbb{Z}^g} e^{2\pi i (z,m) + \pi i (Bm,m)}$$

$$z = (z_1, \dots z_g)$$

$\theta(z)$ is holomorphic function of z

- $\theta(z + e_i) = \theta(z)$
- $\theta(z + B_k) = e^{-2\pi i z_k - \pi i B_{kk}}\,\theta(z)$

- $\theta(z+e_i) = \theta(z)$
- $\theta(z+B_k) = \underbrace{e^{-2\pi i z_k - \pi i B_{kk}}}\,\theta(z)$

Ex $g=1$

$$\theta(z) = \sum e^{2\pi i m z + \pi i m^2 \tau} \qquad B_{11} = \tau$$

$$\theta(z+\tau) = \sum e^{2\pi i\, m(z+\tau) + \pi i m^2 \tau}$$

$$\sum e^{2\pi i m z + \pi i (m^2 \tau + 2\, m\tau + \tau) - \pi i \tau}$$

$$m \to m+1 \qquad \left(\sum e^{2\pi i (m+1) z + 2\pi i (m+1)^2 \tau}\right) - \pi i \tau - 2\pi i z$$

Abel map $\Gamma \to \mathcal{J}(\Gamma)$

$p \in \Gamma \qquad A_k(p) = \int_q^p \omega_k$

$A(p)$

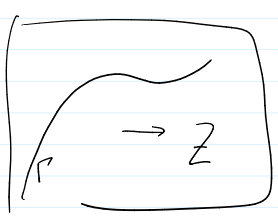
Jacobi inversion theorem

Consider multivalued function on Γ

$$F(p) := \theta(A(p) + Z)$$

zeros of F are well-defined !!

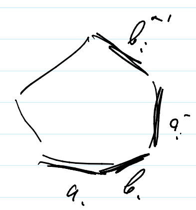
- If F is not identically zero then



$\mathcal{J}(\Gamma$ $\qquad \theta(Z) \neq 0$

Then
- # zeros p_s of $F(p_s) = 0$ $= g$
- $Z = -\sum_{s=1}^{g} A(p_s) + K$

$$S^g(\Gamma) \approx \mathcal{J}(\Gamma)$$



Proof $N = \frac{1}{2\pi i} \oint_{\partial\Gamma} \frac{d\theta(A(p)+Z)}{\theta(A(p)+Z)} = g$

$$\sum A_k(p_s) = \frac{1}{2\pi i}\int A_k(p)\, d\ln \theta(A(p)+Z) \overset{?}{=} -Z + K$$

$$A_k(p) = \int_{q_0}^{p} \omega_k$$

Let $p_1, \dots p_g$ be zeros of

$F(p) = \theta(A(p) + Z)$

Let $\gamma_1, \dots \gamma_g$ be zeros of

$$F(p) = \theta(A(p) + Z)$$

- $$\theta(A(p) \underbrace{- A(Q) + A(\gamma_1)}) + Z) = F_Q(p)$$

Claim Zeros of F_Q are $Q, \gamma_2, \dots \gamma_g$

$$Z + A(\gamma_1) - A(Q) = -\sum_{s=1}^{g} A(\gamma_s) + A(\gamma_1) - A(Q) + K$$

$\underline{Ex}$ Write a basic of functions with poles at $\gamma_1, \dots, \gamma_g, \gamma_{g+1}$

$$f(p) = \frac{\theta(A(p) + A(\gamma_g) - A(q) + Z)\,\theta(A(p) + Y)}{\theta(A(p) + Z)\,\theta(A(p) - A(\gamma_{g+1}) + A(\gamma_g) + Z)}$$

$f(q) = 0$ $\quad \gamma_1 \dots \gamma_g, \gamma_{g+1}$

$\gamma_1 \dots \gamma_{g-1}, \gamma_{g+1} \qquad \gamma_1 \dots \gamma_{g-1}$

$$Y + A(\gamma_g) - A(q) + Z = Z + Z - A(\gamma_{g+1}) + A(\gamma_g)$$

$$Y = Z + A(q) - A(\gamma_{g+1}) \qquad \left(-A(\gamma_{\dots}) - A(\gamma_g) + A(q) - A(\gamma_{g+1})\right)$$

$\underline{\underline{Ex}}$ Let $\mathcal{D} = \gamma_1 + \dots + \gamma_{g+r-1}$ $\quad P_1 \dots P_r$

Write $r_i(p)$ with $\gamma_1 \dots \gamma_{g+r-1}$

normalized $\quad r_i(P_j) = \delta_{ij}$.