# Lecture 4

Thursday, October 1, 2020 8:57 AM

## Recall:

$$L(z) = u_0 + \sum_{i=1}^{N} \frac{u_i}{z - z_i} \qquad u_0^{ij} = u_0^i \delta^{ij}$$

Spectral transform

$$L \iff \underset{\text{spectral curve}}{\Gamma}, \quad \mathcal{D}\text{ - divisor of degree } g+r-1$$

$$g = N\frac{r(r-1)}{2} - (r-1) \qquad \det(k\cdot 1 - L(z)) = 0$$

Inverse transform

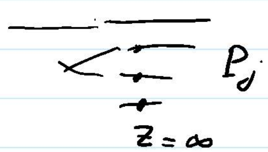
$p \in \Gamma = (k, z)$

$L(z)\psi(p) = k(z)\psi(p)$

Riemann-Roch

$$h^0(\mathcal{D}) = \dim \mathcal{L}(\mathcal{D}) = \deg \mathcal{D} - g + 1 = r$$

$$\psi_i \in \mathcal{L}(\mathcal{D}) \qquad \psi_i(P_j) = \delta_{ij}$$




$$L = \hat{\Psi}(z)\,\hat{k}(z)\,\hat{\Psi}^{-1}(z)$$

$$\hat{\Psi}(z) = \Big( \psi(k_j, z) \Big) \qquad k_j(z)$$

$L$ does not depend on an ordering of $k_j(z)$ $\Rightarrow$ it is a meromorphic function

Ex. Easy to show that it has simple poles at $z_i$.

Next goal is to solve the Lax eq-ns

## Next goal is to solve the Lax eq-ns

$$\partial_\alpha L = [A_\alpha(z), L(z)] \quad \text{Lax equations}$$

$$A(z) = \frac{v}{z-\mu}$$

$$\partial_\alpha u_0 = 0 \qquad \partial_\alpha u_i = \left[\frac{v}{z_i - \mu}, u_i\right] \qquad [v, L(\mu)] = 0$$

$$v = L^n(\mu) \qquad \alpha = (n, \mu)$$

For different choices of $\alpha$ $\partial_\alpha$ commute

$$L(t,z) \qquad A(t,z)$$

- $\partial_t \Gamma = 0 \qquad \mathcal{D}(t) = ?$

$$\begin{cases} L(t,z)\psi(t,p) = k(z)\psi(t,p) \\ (\partial_t - A)\psi(t,p) = \psi(t,p) f(t,p) \end{cases}$$

$$[(\partial_t - A), L] = 0$$

$f(t,p)$ is a scalar function on $\Gamma$

$\psi(t,p)$ has poles at $\gamma \in \mathcal{D}(t)$

$$\psi(t,p) = \frac{\varphi(t)}{z - z(\gamma(t))} + O(1) \qquad \odot^{z}_{\gamma(t)}$$

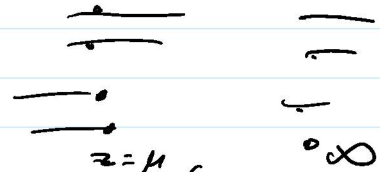
$$\partial_t \psi = \frac{\varphi(t)\,\dot{z}(\gamma(t))}{(z - z(\gamma(t)))^2} + O\left((z - z(\gamma(t)))^{-1}\right)$$

$$\Rightarrow \quad f(t,p) = \frac{\dot{z}(\gamma(t))}{(z - z(\gamma(t)))} + O(1)$$

$$(z - z(\gamma(t)))$$

$$\Psi(t,p) = \psi(t,p)\, \underline{\exp\left(-\int_0^t f(t',p)\,dt'\right)}$$

$$\begin{cases} \underline{(\partial_t - A)\,\underline{\Psi}} = 0 \\ L\underline{\Psi} = \kappa \Psi \end{cases}$$

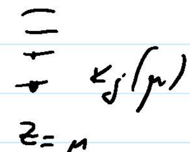



- $\underline{\Psi}(t,p)$ is meromorphic on $\Gamma \setminus (\kappa_j(\mu), \mu)$ with poles at $\mathcal{D}(0)$ only

$$e^{-\int_0^t f(t',p)\,dt'} = e^{\left(\ln(z - z(\gamma(t)))\right) + O(1)} = (z - z(\gamma(t)))\,O(1)$$

- Near $\mu$

$$\left(\partial_t - \frac{v(t)}{z-\mu}\right)\psi = \psi\, f$$




$$\psi = \psi_j + O(z-\mu)$$

$$f = \frac{\nu_j}{z-\mu} + O(1) \qquad \nu_j =$$

$$v(t)\,\underline{\underline{\psi_j}} = \underline{\underline{\psi_j}}\,\nu_j \qquad v = L^n(\mu)$$

$$\nu_j = \kappa_j^n \text{ - does not depend on } t$$

$$-\int_0^t f(t',p)\,dt' = \frac{-\nu_j t}{z-\mu} + O(1) \qquad p = (\kappa_j, \mu)$$

- $\underline{\Psi}(t,p)$ is meromorphic on $\Gamma \setminus \{(\kappa_j, \mu)\}$ with poles at $\mathcal{D}(0)$

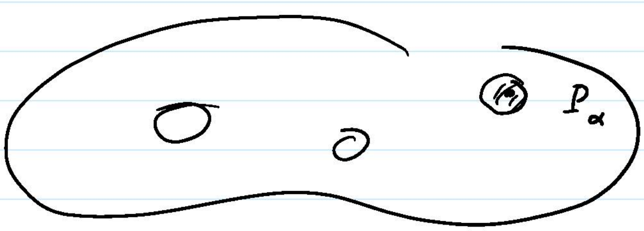
- In the neighborhood of $(k_j, \mu)$ has essential singularity of the form

$$\underline{\Psi} = e^{-\frac{v_j(t)}{z-\mu}} \left( \sum_{s=0}^{\infty} \xi_s(t)(z-\mu)^s \right) \qquad \mathcal{D}(0)$$

$$L(t) = \underline{\hat{\Psi}}(t,z)\, \hat{K}(z)\, \hat{\Psi}^{-1}(t,z)$$

## Baker-Akhiezer functions

$\Gamma$ smooth genus $g$ algebraic curve with fixed local coordinate $z_\alpha$ near $P_\alpha$




$z_\alpha(P_\alpha) = 0$

$q_\alpha(k)$ a set of polynomials

$$q_\alpha(k) = \sum_{i=1}^{\infty} \underline{\underline{t_{\alpha i}}}\, k^i \qquad t = (t_{\alpha i})$$

Given divisor $\mathcal{D}$ define

$\mathcal{L}_q(\mathcal{D})$ is the space of functions on $\Gamma \setminus \{P_\alpha\}$ that are

- meromorphic on $(\Gamma \setminus \{P_\alpha\})$ with poles at $\mathcal{D}$ of multiplicity not bigger than multiplicity $\mathcal{D}$

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- $\psi \in \mathcal{L}_q(\mathcal{D})$ near $P_\alpha$ has the form

$$\psi(t, z_\alpha(p)) = e^{q_\alpha(z_\alpha^{-1})} (\text{regular function}), \text{ i.e.}$$

$$= e^{\sum_{i=1}^{\infty}(t_{\alpha i} z_\alpha^{-i})} \left( \sum_{s=0}^{\infty} \xi_{s\alpha}(t) z_\alpha^s \right)$$

$t = (t_{\alpha,i})$

<u>Lemma</u> For a generic (non-special) $\mathcal{D}$

$$|t| \ll 1 \qquad \dim \mathcal{L}_q(\mathcal{D}) = \deg \mathcal{D} - g + 1 \qquad \deg \mathcal{D} \geq g$$

- The divisor of zeros of $\psi(t, p)$ is of degree $|\mathcal{D}|$

<u>Consider</u> $\frac{d\psi}{\psi}$. It is meromorphic <u>differential</u>

$$\frac{d\psi}{\psi} = dq_\alpha(z) + O(1)$$

$$\operatorname{res}_{P_\alpha} \frac{d\psi}{\psi} = 0 \implies$$

$$\#\{\text{poles}\} = \#\{\text{zeros}\}$$