# Lecture 3

Thursday, September 24, 2020 8:34 AM

## Recall

We considered as one of the examples of "phase" spaces of integrable systems

$$L(z) = \underset{\frown}{u_0} + \sum_{i=1}^{N} \frac{u_i}{z - z_i} \qquad u_0, u_i \in GL_r$$

$$u_0^{ij} = u^i \delta_{ij} \qquad u_i \neq u_j$$

On the space of "$L$" we defined $A(L)$ s.t

$\underline{\dot{L} = [A, L]}$ is well-defined system of eq-ns

$$\dot{u}_0 = 0 \qquad \dot{u}_i = f(u_0, \dots u_N)$$

We saw that $A(L)$ is not unique but for all choices of $A$ the flows commute

## Spectral transform

$$L(z) \to R(k, z) = \det(k \cdot \mathbb{1} - L(z))$$

$$= k^r + \sum r_i(z) k^{r-i}$$

$r_i(z)$ rational functions of $z$ with poles at $z_i$ of order $i$

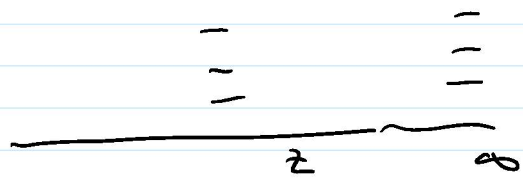
$$r_i(z) = \sum^{N} \sum^{r} \frac{r_{ij}}{\cdots} \dots + \rho_i$$

$$r_i(z) = \sum_{i=1}^{N} \sum_{j=1}^{r} \frac{r_{ij}}{(z-z_i)^j} + \rho_i$$

\# of parametes $r_{ij}, \rho_i$ =

$$= N \frac{r(r+1)}{2} + r$$

$$\Gamma \in C^2 \qquad \det(\kappa \cdot 1 - L(z)) = 0$$



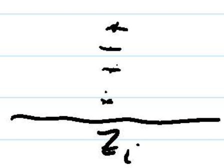

For generic $r_{ij}, \rho_i$ the curve is smooth of genus

$$2g-2 = \underline{\underline{\nu}} - 2r \quad \text{Riemann-Hurwitz}$$

$\nu$ = # branch points = # of zeros on $\Gamma$

$\partial_\kappa R(p) \qquad p = (\kappa, z) \in \Gamma$

= # of poles $= N\, r(r-1) = \nu$




$$\kappa(z) \sim \frac{v_{ij}}{z - z_i}$$

z

$\partial_\kappa R$ has a pole of order $r-1$ in the preimages of $z = z_i$

$$g = N\frac{r(r-1)}{2} - (r-1)$$

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$L(z) \Longleftrightarrow \Gamma, \mathcal{D}$ - divisor of degree $g+r-1 = N\frac{r(r-1)}{2}$

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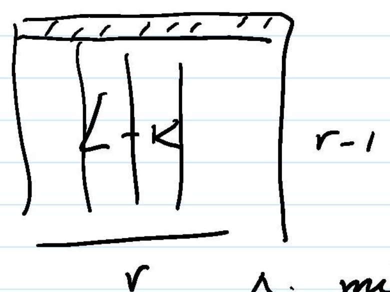
$$p = (k, z) \in \Gamma$$

$$L(z)\psi(p) = k\,\psi(p)$$

$\psi(p)$ normalized eigenvector

$$\sum \psi_i = 1 \quad \text{or} \quad \psi_1 = 1$$

$$\psi_i = \frac{\Delta_i}{\sum_{j=1}^{r} \Delta_j}$$




$\Delta_i$ minor

$\mathcal{D}$ = divisor of poles of $\psi$

- $\deg \mathcal{D} = g + r - 1 = \frac{r(r-1)}{2}$

<u>Proof</u> Consider

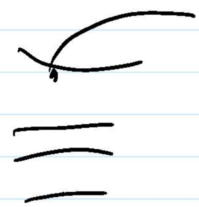
$$F(z) = \det{}^2\left(\hat{\Psi}(z)\right)$$

$$\hat{\Psi}(z) = \left(\psi(k_1(z), z)\ \psi(k_2(z), z)\ \dots\right)$$

$$\Psi(z) = \begin{pmatrix} \psi(\kappa_1(z), z) & \psi(\kappa_2(z), z) & \dots \end{pmatrix}$$

Poles of $F(z)$ are projections of poles of $\psi$ $\Rightarrow$

$\#$ poles of $F = 2 \deg \mathcal{D} = \nu$



$$\psi = \psi_0 + \psi_1 \sqrt{z-e}$$

$$\begin{pmatrix} \psi_0 + \psi_1\sqrt{z-e} & \psi_0 - \psi_1\sqrt{z-e} & \cdots \end{pmatrix} \quad z$$

$L(z) \longrightarrow \Gamma, \mathcal{D}$ direct spectral transform

Inverse spectral transform

Given $\Gamma, \mathcal{D}$ deg $g+r-1$

Consider $\mathcal{L}(\mathcal{D})$ the space of meromorphic functions on $\Gamma$ with poles at $\mathcal{D}$ s.t. $(f)+\mathcal{D} \geq 0$

Riemann-Roch for a generic $\mathcal{D}$

$$\dim \mathcal{L}(\mathcal{D}) = h^0(\mathcal{D}) = \deg \mathcal{D} - g + 1$$

$$\dim \mathcal{L}(\mathcal{D}) = h(\mathcal{D}) = \deg \mathcal{D} - g + 1$$
$$= g + r - 1 - g + 1 = r$$

$\psi_i \in \mathcal{L}(\mathcal{D}) \qquad \psi_i(P_j) = \delta_{ij}$

$P_r$

$P_1$

$z = \infty$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_r \end{pmatrix}$$

$\psi(p)$

- $\exists!\ L(z)$ s.t. $L(z)\psi(p) = \kappa(p)\psi(p)$

$$\underline{u_0^{ij} = u^i \delta_{ij}}$$

$\kappa(P_j) = u^i$

Define $u_i$ from the equation

$z = 0$

$$u_i V_i = V_i \hat{v}_j \qquad u_i = V_i \hat{v}$$

$$V_i = \hat{\Psi}(z_i) \qquad k \sim \frac{v_{ij}}{z - z_j}$$

$z = z_i$

$$L(z) = \hat{\Psi}(z)\, \hat{k}(z)\, \hat{\Psi}^{-1}(z)$$

- do not depend on ordering of sheets in the definition of $\hat{\Psi}$

the definition of $\Psi$

$$L\hat{\Psi} = \hat{\Psi}\hat{k}$$

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$$\tilde{L}(z)/GL_r = \left(\Gamma, [\mathcal{D}] \in J(\Gamma)\right)$$

$\mathcal{D} \sim \mathcal{D}'$ if $\exists f$ with poles at $\mathcal{D}$ and zeros at $\mathcal{D}'$

$$\hat{f} \quad \psi(p) = \psi'(p) f(p)$$

$$\hat{f} = \mathrm{diag}\left(f(p_i)\right)$$

$$L' = \hat{f} L \hat{f}^{-1}$$

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$$, = \Gamma, \mathcal{D}(t)$$