# Lecture 2

Thursday, September 17, 2020 9:58 AM

<u>Recall</u>:

Integrable system $\iff$ $\boxed{\text{compatibility conditions of overdetermined system of linear problems}}$

Vast majority of examples

Ex Lax equation

$$\dot{L} = [A, L] \qquad \begin{matrix} L(t) \\ A \end{matrix} \text{ linear operator}$$

KdV a) $\quad u_t + \frac{3}{2} u u_x - \frac{1}{4} u_{xxx} \qquad L = -\partial_x^2 + u(x,t)$

NLS b) $\quad i r_t = r_{xx} \pm |r|^2 r \qquad L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x + \begin{pmatrix} 0 & r \\ \pm \bar{r} & 0 \end{pmatrix}$

Toda c) $\quad \ddot{x}_n = e^{x_{n-1} - x_n} - e^{x_n - x_{n+1}}$

$$(L\psi)_n = \psi_{n+1} + v_n \psi_n + \psi_{n-1}$$

$$\psi = (\psi_n)$$

Zero-curvature equations

$$0 = [\partial_\xi - U, \partial_\eta - V]$$

$$U = \begin{pmatrix} v & 1 \\ \lambda^{-1} & -v \end{pmatrix} \qquad V = \begin{pmatrix} 0 & \lambda e^u \\ e^{-u} & 0 \end{pmatrix}$$

$$2u_{\xi\eta} = e^u - e^{-u} \qquad u \to iu$$

d) <u>sine-gordon</u> $\qquad u_{\xi\eta} = \sin u$

e) $(L, A, B)$ <u>triple</u>

$$[L, \partial_t - A] = BL$$

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$$L = \partial\bar\partial + u \quad , \quad A = \partial^3 + v\,\partial \qquad\qquad \partial = \partial_z \quad \bar\partial = \partial_{\bar z}$$

$$\dot L = [A, L] + BL$$

$$[\underline{\underline{\partial^3}} + \underline{v\partial}, \underline{\partial\bar\partial} + \underline{\underline{u}}] = 3u_z\partial^2 + 3u_{zz}\partial + u_{zzz}$$

$$- v_{\bar z}\partial^2 - v_z\partial\bar\partial - v_{z\bar z}\partial + v u_z$$

$$\dot u = \cancel{3u_z\partial^2} + \cancel{3u_{zz}\partial} + u_{zzz} - \cancel{v_{\bar z}\partial^2} - \cancel{v_z\partial\bar\partial} - \cancel{v_{z\bar z}\partial} + v u_z$$

$$B(\cancel{\partial\bar\partial} + u) \qquad = u_{zzz} + v u_z + v_z u$$

$$3u_z = v_{\bar z} \qquad B = v_z \qquad = u_{zzz} + (vu)_z$$

Equation Novikov-Veselov

$$L\psi = 0 \qquad\qquad (\dot L - [AL])\psi = 0$$

$$(\partial_t - A)\psi = 0 \qquad\qquad =$$

If there are enough $\psi$ $\Rightarrow$ Lax equation

If the ideal of operators annihilating $\psi$ is generated by $L$ $\Rightarrow$ $L, A, B$ triple

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!!! Linear operators depend on a "spectral parameter" (explicite or latent form)

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Ex Toda lattice

$$\ddot x_n = e^{x_{n-1} - x_n} - e^{x_n - x_{n+1}} \qquad x_{n+N} = x_n + l$$

$N$ periodic Toda lattice

$$(L\psi)_n = \psi_{n+1} + v_n\psi_n + c_n\psi_{n-1} \qquad v_n = v_{n+N}$$

$$c_n = e^{x_n - x_{n-1}} = c_{n+N}$$

$$(L\psi)_n = \psi_{n+1} + v_n\psi_n + c_n\psi_{n-1} \qquad v_n = v_{n+N}$$

$$c_n = e^{x_n - x_{n-1}} = c_{n+N}$$

$$\underline{\psi_{n+N} = w\,\psi_n}$$

$$\begin{pmatrix}\psi_1\\ \psi_2\\ \vdots \\ \psi_N\end{pmatrix} \qquad \psi_{N+1} = w\,\psi_1 \;\ldots \qquad \psi_0 = w^{-1}\psi_N$$

$$\begin{pmatrix} \ddots & & \\ c_n & v_n & 1 \\ & & \ddots \end{pmatrix}\begin{pmatrix}\vdots\\ \psi_0\\ \psi_1\\ \psi_2\end{pmatrix} \qquad L(w) = \begin{pmatrix} v_0 & 1 & & w^{-1}c_N \\ & \ddots & & \\ & c_n\; v_n & 1 & \\ w\,c_N & & & \ddots \end{pmatrix}$$

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## Next goal is to define and then "solve"

- Phase space

$\boxed{L}$ space of operators $\qquad [A, L]$ - vector field

### Two basic examples

① $L(z)$ - matrix rational functions

$$L(z) = \sum \frac{u_i}{z - z_i} + u_0 \qquad u_0, u_i \in \mathrm{Mat}_{r\times r}$$

Particular case of the Hitchin system

- Lax equation $\Rightarrow$ integrals of motion
- Lax equation $\Rightarrow$ "defines" $A$

$$\det\left(k\cdot \mathrm{Id} - L(z)\right) = k^r + \sum_{i=1}^{r} s_i(z)\,k^{r-i}$$

$s_i(z)$ - rational function of $z$

$$\dot{s}_i(z) = 0$$

$\dot{s}_i(z) = 0$

Consider $\left(\partial_t - A\right)\Psi(z,t) = 0 \quad \Psi(z,0) = 1$

$$\left(\partial_t - A\right) L(t)\Psi(z,t) = 0$$

$$L(t)\Psi(t) = \Psi(t)\, G$$

$$t=0 \Rightarrow G = L(0)$$

$$L(t) = \Psi(t)\, L(0)\, \Psi^{-1}(t)$$

$\#\{\text{integrals}\}$

$s_i(z)$ has poles of order $i$ at $z_i$

$$N\left(\sum_{i=1}^{r} i\right) + r = N\,\frac{r(r+1)}{2} + r$$

$$\dim\left(L(z)\right) = N r^2 + r^2$$

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Lax equations are invariant under gauge transform

$$L \to g L g^{-1} \qquad A = g A g^{-1} \quad g\text{-const}$$

$$\dim\left(L/_{GL_r}\right) = N r^2 + 1$$

Consider
$$L(z) = \sum \frac{u_i}{z - z_i} + u_0 \qquad u_0^{ij} = u_0^i \delta_{ij}$$

For fixed $u_0$ $\dim L = N r^2 + r$

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Equations

$\dot{L} = [A, L]$ $\quad [A,L]$ - should be a rational function with the same divisor of poles rational function with the same divisor of poles

$$A = \frac{v}{z-\mu} \longrightarrow [v, L(\mu)] = 0$$

$$\mu \neq z_i \qquad v = L^n(\mu)$$

$$\partial_{t_{n,\mu}} u_i = \left[ \frac{L^n(\mu)}{z_i - \mu}, u_i \right] \qquad u_i, \quad \dot{u}_0 = 0$$

- $\left[ \partial_{t_{n,\mu}}, \partial_{t_{n',\mu'}} \right] = 0$

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$$[\partial_\alpha - A_\alpha, \partial_\beta - A_\beta] = 0 \qquad \alpha = (n, \mu)$$

$$\beta = (n', \mu')$$

$$\partial_\beta A_\alpha - \partial_\alpha A_\beta + [A_\alpha, A_\beta] = 0$$

$$\partial_{t_{n',\mu'}} \underline{L^n(\mu)} + \left[ \underline{L^n(\mu)}, \frac{L^{n'}(\mu')}{\mu - \mu'} \right] = 0$$

$$\partial_{t_{n',\mu'}} L^n(z) = \left[ \frac{L^{n'}(\mu')}{z - \mu'}, L^n(z) \right] \Big|_{z=\mu}$$

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- Lax equation by itself "defines" $A = A(L)$ (ambiguity in the definition leads to commuting flows)

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<u>Ex</u> <u>Rigid body</u>

$$L = z a + U \qquad a^{ij} = a_i \delta_{ij} \qquad a^i = \underline{\underline{const}}$$

$$A = z b + V$$

$$\dot{U} = \dot{L} = [z b + V, z a + U] = z^2 [b, a] + z \underline{([V, a] + [b, U])}$$

$$\dot U = \dot L = [zb+V,\ za+U] = z^2[b,a] + z\underline{([V,a]+[b,U])} + [V,U]$$

$$[b,a]=0 \qquad b^{ij} = b_i\,\delta_{ij} \qquad a_i \neq a_j$$

$$V^{ij}(a_i-a_j) = (b_i-b_j)\,U^{ij}$$

$$V^{ij} = \frac{b_i-b_j}{a_i-a_j}\,U^{ij} \qquad b_i = a_i^2$$

$$V^{ij} = (a_i+a_j)\,U^{ij} \qquad V = aU+Ua$$

$$\dot U = [aU+Ua,\ U]$$

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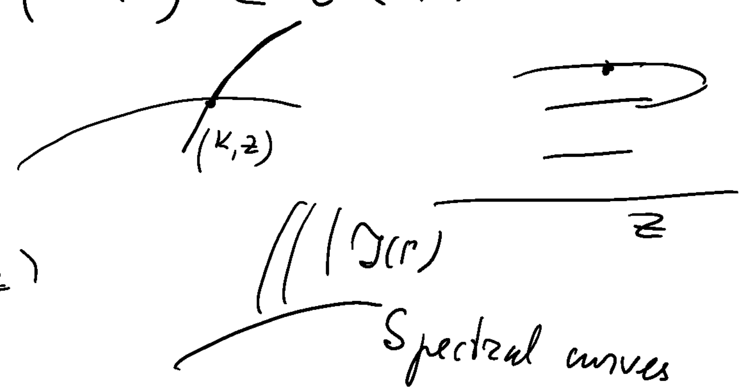
Next step $\to$ spectral transform

$L(z) \to$ (spectral curve $\Gamma$, ? line bundle of eigenvectors)

$$0 = \det(k - L(z)) \subset \mathbb{C}^2(k,z)$$




$L(z)$