Lecture 1

Thursday, September 10, 2020 6:57 AM

10/09/20

# Integrable systems

## I. Course plan

Solitons equations and associated structures

Ex ✓ ① $4u_t - 6uu_x + u_{xxx} = 0$ Korteweg-de Vries KdV

$u(x,t)$

✓ ② $i\psi_t = \psi_{xx} \pm |\psi|^2\psi$ Non linear Schrodinger NLS

✓ ③ $u_{tt} - u_{xx} = \sin u$ sine-gordon

✓ ④ $\ddot{x}_n = e^{x_n - x_{n-1}} - e^{x_{n+1} - x_n}$ Toda lattice

✓ ⑤ $\ddot{x}_n = 2\sum_{m\neq n} \frac{-g}{(x_n - x_m)^3}$ Calogero-Moser

$V(x_n - x_m)$

rational, trigonometric, elliptic

$V(x) = \frac{1}{x^2}, \ \frac{1}{sh^2(x)}, \ \wp(x)$.

✓ ⑥ Hitchin system

Rough classification

$\underline{(0+1)}$, $\underline{(1+1)}$ $\underline{(2+1)}$ $u(x,y,t)$

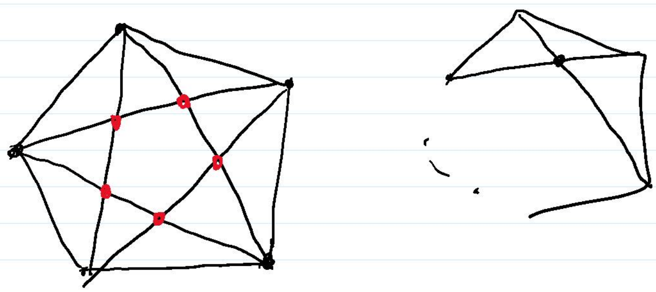
⑦ $\pm 4\frac{\partial^2 u}{\partial y^2} = (4u_t - 6uu_x + u_{xxx})_x$ KP

⑧ $(\partial_t^2 - \partial_x^2)\varphi_n = e^{\varphi_n - \varphi_{n-1}} - e^{\varphi_{n+1} - \varphi_n}$ 2D Toda

$\varphi_n(t,x)$ $(n,x)$

Relatively recently (0+0) systems (discrete time)

Ex Pentagram map



## II. Spectral transform of linear operators

## III Solutions (Baker-Akhiezer functions)

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IV Whitham perturbation theory

(V)? Applications

- Geometry of moduli space of curves
- TQFT (a.k.a associativity eq-ns, Frobenious manifolds)
- Seiberg-Witten solution of $N=2$ SUSY
- Conformal maps

<u>Prerequisits</u> (beginners level of algebraic geometry, Hamiltonian theory)

"Def" Solition equation (Integrable systems) $\Longleftrightarrow$ compatibility condition of overdetermined system of Linear problems

(A) Lax equation

linear operator

$$\begin{cases} L\psi = E\psi \\ (\partial_t - A)\psi = 0 \end{cases} \Longrightarrow [\partial_t - A, L]\psi = 0$$

If there are 'enough' $\psi$ solving the system

$$0 = [\partial_t - A, L] \iff \underline{\dot{L} = [A, L]}$$

Ex $\qquad L = -\partial_x^2 + u(x,t)$

$$A = \partial_x^3 + v\,\partial_x + w \qquad \begin{matrix} v(x,t) \\ w(x,t) \end{matrix}$$

$\underline{\dot{L} = \dot{u}}$

$\surd \quad [-\partial_x^2 + u, \partial_x^3 + v\partial_x + w] = -2v_x\partial_x^2 - v_{xx}\partial_x$

$$-3u_x\partial^2 - 3u_{xx}\partial - u_{xxx} - vu_x - 2w_x\partial - w_{xx} = -\dot{u}$$

$$2v_x + 3u_x = 0 \qquad v = \frac{3}{2}u + c \qquad c = 0$$

$$v_{xx} + 3u_{xx} + 2w_x = 0$$

$$-\frac{3}{2}u_{xx} + 3u_{xx} \mp 2w_x = 0 \qquad w = -\frac{3}{4}u_x + c_1^{\,=0}$$

$$-\dot{u} = -u_{xxx} - \frac{3}{2}uu_x + \frac{3}{4}u_{xxx} \Rightarrow$$

$\surd \quad \dot{u} = \frac{3}{2}uu_x - \frac{1}{4}u_{xxx} \qquad$ KdV equation

① KdV $\quad \longleftrightarrow \quad L = -\partial_x^2 + u(x)$

② NLS $\quad L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\partial_x + \begin{pmatrix} 0 & \psi \\ \pm\bar{\psi} & 0 \end{pmatrix}$

$$A = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\partial_x^2 + \frac{1}{2}\left[\begin{pmatrix} 0 & \psi \\ \pm\bar{\psi} & 0 \end{pmatrix}\partial_x + \partial_x\begin{pmatrix} 0 & \psi \\ \pm\bar{\psi} & 0 \end{pmatrix}\right]$$

$$+ \begin{pmatrix} \pm|\psi|^2 & 0 \\ 0 & \mp|\psi|^2 \end{pmatrix}$$

$$i\psi_t = \psi_{xx} \pm |\psi|^2 \psi$$

4 $$\begin{cases} \psi_{n+1} + v_n \psi_n + c_n \psi_{n-1} = E\psi_n \\ \partial_t \psi_n = c_n \psi_{n-1} \end{cases} \qquad L\psi = E\psi$$

$$c_n = e^{x_{n-1} - x_n} \qquad v_n = \dot{x}_n$$

$$\boxed{\ddot{x}_n = e^{x_{n-1} - x_n} - e^{x_n - x_{n+1}}}$$

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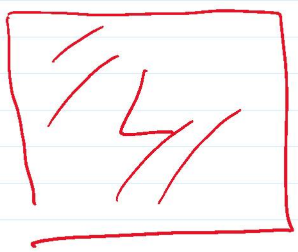
Phase space of an integrable system is the space of operators (depending on a spectral parameter)



$$\underset{\sim}{u(x)} \longleftrightarrow \underset{\sim}{-\partial^2 + u} = L$$

$$\longleftrightarrow \underline{(L\psi)_n = \psi_{n+1} + v_n \psi_n + c_n \psi_{n+1}}$$

$$c_n = e^{x_{n-1} - x_n}, \quad v = p_n = \dot{x}_n$$

Basic $\mathcal{E}x$ $$\boxed{L(z) = \sum_{i=1}^{m} \frac{u_n}{z - z_i} + u_0}$$ $z$ - the spectral parameter

$m, \; z_i, \quad u_i \in Mat_r(\mathbb{C})$

$u_0, u_i$

$\mathcal{E}x$ $\boxed{L = -\partial^2 + u(x)}$ Q: where is

Ex $\boxed{L = -\partial_x^2 + u(x)}$ Q: where is a spectral parameter

Suppose $u(x+1) = u(x)$

Then $L$ can be restricted to the space of functions s.t. $f(x+1) = \underline{\underline{z}}\, f(x)$

$$L\Big|_{\{f(x+1) = z f(x)\}} = L(z)$$