# Review of the ADHM construction

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## 1 Introduction

Here we review the ADHM construction, that is the construction of the $SU(n)$ bundle over $S^4$ with (anti-)self-dual connection and given instanton number $k$. The construction is motivated by the twistor map $\pi : \mathbb{C}P^3 \to S^4$ which can be viewed as the way to consider simultaneously all possible complex structures on the tangent fibers to a sphere. This interpretation allows to address the (anti-)self-dual forms as the forms with special complex coordinates dependence.

## 2 Basic example: $SU(2)$ $k = 1$ instanton

The first example to be considered is $SU(2)$ instanton with instanton number $k = 1$. This instanton appears in tautological bundle over $S^4$. The most natural formalism to describe it is quaternions since

$$S^4 \simeq \mathbb{H}P^1 = \{(x : y) \in \mathbb{H}^2 : (\lambda x : \lambda y) \sim (x : y)\} \tag{1}$$

Tautological bundle naturally arises as a bundle over $\mathbb{H}P^1$ where the slice over a point is the line that represents this point in $\mathbb{H}^2$. It is fully described by the following data

$$\begin{aligned} \varphi_0(x,t) &= \frac{(tx, t)}{\sqrt{1+|x|^2}} \text{ over } (x : 1) \in U_0 \\ \varphi_\infty(x,t) &= \frac{(t, tx)}{\sqrt{1+|x|^2}} \text{ over } (1 : x) \in U_\infty \end{aligned} \tag{2}$$

The maps $\varphi_0(x,t)$, $\varphi_1(x,t)$ introduce the local coordinates on the corresponding open sets. To find the transition function between them we pick a point of the bundle that has the coordinates $(x,t)$ under $\varphi_0$ and compute its coordinates under $\varphi_1$. The $x$ coordinate is clearly gets inverted and the $t$ coordinate follows from the relation

$$\frac{(tx,t)}{\sqrt{1+|x|^2}} = \frac{(tx, txx^{-1})}{|x|\sqrt{1+|x^{-1}|^2}} \tag{3}$$

Therefore the transition function is

$$g = g_{0\infty} = \frac{x}{|x|} \tag{4}$$

Note that it acts from the right on a section. The next step is to specify the connection which we now do ad hoc for the needs of the example.

$$\mathcal{A} = \Im\Big\{\frac{xd\bar{x}}{1+|x|^2}\Big\} = \frac{1}{2}\frac{xd\bar{x} - dx\bar{x}}{1+|x|^2},\ x \in U_0 \tag{5}$$

Note that it has been transposed to stick to the traditional notation where the connection acts from the left. To verify that this is indeed a globally defined connection we do the gauge transformation:

$$\begin{aligned} g^{-1}\Im\Big\{\frac{xd\bar{x}}{1+|x|^2}\Big\}g + g^{-1}dg = \Im\Big\{x^{-1}\frac{-dx\bar{x}}{1+|x|^2}x + x^{-1}dx\Big\} = \\ \Im\Big\{\frac{x^{-1}dx}{1+|x|^2}\Big\} = \Im\Big\{\frac{x^{-1}d(\bar{x}^{-1})}{1+|x|^{-2}}\Big\},\ x^{-1} \in U_\infty \end{aligned} \tag{6}$$

So the connection in the vicinity of $\infty$ is not only correctly defined but also given by the same formula. It remains to compute the curvature which is straightforward:

$$F = d\mathcal{A} + \mathcal{A}\wedge\mathcal{A} = \frac{dx\wedge d\bar{x}}{(1+|x|^2)^2} \tag{7}$$

It is self-dual:

$$\begin{aligned} dx\wedge d\bar{x} = -2i(dx_1\wedge dx_2 + dx_3\wedge dx_4) - 2j(dx_1\wedge dx_3 - dx_2\wedge dx_4) \\ - 2k(dx_1\wedge dx_4 + dx_2\wedge dx_3) \end{aligned} \tag{8}$$

The instanton number follows from the following integral

$$k = -\frac{1}{8\pi^2}\int_{S^4} tr(F\wedge F) = \frac{\text{Vol } S^3}{8\pi^2}\int_0^\infty \frac{48r^3dr}{(1+r^2)^4} = 1 \tag{9}$$

# 3 Unitary bundles over $\mathbb{C}P^3$ and $S^4$

We now turn to the general $SU(n)$ $k$-instanton. First, we construct a unitary bundle over $\mathbb{C}P^3$ which turns out to be closely related to the original problem. In the construction we use the following

- $\sigma : \mathbb{C}P^3 \to \mathbb{C}P^3, \sigma(z_1 : z_2 : z_3 : z_4) = (-\bar{z}_2 : \bar{z}_1 : -\bar{z}_4 : \bar{z}_3)$

  The map $\sigma$ is anti-involution which distinguishes the lines in $\mathbb{C}P^3$ which are preserved by it. These lines are called real lines (although they are actually 2-spheres as real manifolds). The real lines are the fibers of the projection

$$\begin{gathered} \pi : \mathbb{C}P^3 \to S^4 \simeq \mathbb{H}P^1 \\ \pi(z_1 : z_2 : z_3 : z_4) = (z_1 + z_2 j : z_3 + z_4 j) \end{gathered} \tag{10}$$

  The real line through a point $z$ is the line through the pair of points $(z, \sigma z)$.

- $W$ - complex vector space, $\dim_{\mathbb{C}} W = k$
- $V$ - complex vector space, $\dim_{\mathbb{C}} V = 2k + n$
- $\langle \ , \ \rangle$ - Hermitian form on $V$
- $A(z) = z_i A^i : W \to V$ - full rank map, $\langle A(z), A(\sigma z)\rangle = 0$ for all $z$

Now the construction of the bundle is straightforward:

$$E(z) = (A(z) \oplus A(\sigma z))^{\perp} \tag{11}$$

It is a unitary bundle over $\mathbb{C}P^3$, $\operatorname{rank} E = n$. By construction, $E$ is trivial over the lines $(z, \sigma z) \subset \mathbb{C}P^3$.

As we already pointed out, $\phi : E \to \mathbb{C}P^3$ is trivial above the fibers of the projection $\pi : \mathbb{C}P^3 \to S^4$. It follows that there exist $\tilde{\pi}$, $\hat{\phi}$ such that the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{\tilde{\pi}} & \hat{E} \\ \big\downarrow{\scriptstyle\phi} & & \big\downarrow{\scriptstyle\hat{\phi}} \\ \mathbb{C}P^3 & \xrightarrow{\pi} & S^4 \end{array}$$

The bundle $\hat{E}$ is a unitary bundle over $S^4$. It remains to derive the anti-self-dual connection.

## 4 Anti-self-dual forms and holomorphicity

In this section we discuss the following statement: the anti-self-dual forms on $S^4$ are the forms that have type $(1,1)$ for all compatible complex structures.

A type of a 2-form is understood in terms of the complex coordinates and is represented by a pair

$$(\#\text{ holomorphic 1-forms, }\#\text{ anti-holomorphic 1-forms}) \tag{12}$$

This statement actually boils down to the representation theory of $U(2) \subset SO(4)$. Taking into account that the statement is local we replace $S^4$ with $\mathbb{R}^4$. The space $\Omega = \Lambda^2\mathbb{R}^4$ of 2-forms on $\mathbb{R}^4$ has dimension 6. As a representation of $SO(4)$ it decomposes into the sum of self-dual and anti-self-dual forms

$$\Omega = \Omega^+ \oplus \Omega^- \tag{13}$$

It cannot be decomposed further since there is no $SO(4)$-invariant 2-form.

The subgroup $U(2) \subset SO(4)$ can be realized as the centralizer of the "standard" complex structure $J_0 \in SO(4)$, $J_0^2 = -1$ (any other complex structure corresponds to a different embedding[1]). In other words, $U(2)$ are those $SO(4)$ transformations which are complex-linear for the given complex structure on $\mathbb{R}^4$. The relation between complex structures and complex coordinates is straightforward. Namely, a given complex structure $J \in SO(4)$, $J^2 = -1$ is conjugate via $C \in SO(4)$ to the standard one:

$$J = C^{-1} J_0 C \tag{14}$$

$$J_0 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ C = \begin{pmatrix} a & b & -c & -d \\ -b & a & -d & c \\ c & d & a & b \\ d & -c & -b & a \end{pmatrix}, \tag{15}$$

[1]To be precise, there are two non-conjugate ways to embed $U(2) \to SO(4)$, and here we focus on the conjugacy class represented by $J_0$, so we consider only "a half" of all possible $SO(4)$ complex structures on $\mathbb{R}^4$. The other conjugacy class can be represented by

$$J_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The complex structures $J_0$ and $J_1$ are conjugated via an improper orthogonal transformation.

where $a^2+b^2+c^2+d^2=1$. The matrix $C$ is defined up to $U(2)$ transformation since $J_0$ commutes with $U(2)$. Once $C$ is fixed one introduces the complex coordinates in $\mathbb{R}^4$ by the following formulas (which just state that $J_0$ corresponds to the standard complex coordinates)

$$\begin{aligned} z_1 &= y_1 + iy_2 \\ z_2 &= y_3 + iy_4, \end{aligned} \tag{16}$$

where

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} a & b & -c & -d \\ -b & a & -d & c \\ c & d & a & b \\ d & -c & -b & a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \tag{17}$$

Excluding $y$'s we get

$$\begin{aligned} x_1 + ix_2 &= (a+ib)z_1 + (c+id)\bar{z}_2 \\ x_3 + ix_4 &= -(c+id)\bar{z}_1 + (a+ib)z_2 \end{aligned} \tag{18}$$

Now we consider $\Omega$, as a representation of $U(2)$

$$\Omega = \Omega^{(2,0)} \oplus \Omega^{(1,1)} \oplus \Omega^{(0,2)} = \Omega^{(2,0)} \oplus \Omega_0^{(1,1)} \oplus \langle \omega_0 \rangle \oplus \Omega^{(0,2)} \tag{19}$$

The 2-form $\omega_0 = dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2$ where $z_1$, $z_2$ are complex coordinates introduced by a complex structure. The representation $\Omega_0^{(1,1)}$ is irreducible (which is evident since there is an element of weight 2, $dz_1 \wedge d\bar{z}_2$) and thus coincides with $\Omega^+$ or $\Omega^-$. The form $\omega_0$ is self-dual. Therefore

$$\Omega_0^{(1,1)} = \Omega^- \tag{20}$$

The conclusion is that anti-self-dual forms are precisely those that have type $(1,1)$ for all compatible complex structures.

# 5 Anti-self-dual connection

Clearly, there is orthogonal projection $\hat{P} : S^4 \times V \to \hat{E}$. The connection $\hat{\mathcal{A}}$ in $\hat{E}$ is naturally constructed via orthogonal projection of the trivial connection in $S^4 \times V$. Its pullback $\mathcal{A} = \tilde{\phi}^*(\hat{\mathcal{A}})$ is also given by orthogonal projection of the trivial connection.

Explicitly, we have an orthonormal basis of sections

$$E(z) = (E_1(z), ..., E_n(z)) \tag{21}$$

The orthogonal projector is given by $P = EE^*$. Suppose we have a column of coordinates $f(z) = (f_1(z), ...f_n(z))^T$ which represents a section. We now compute the covariant derivative

$$(d + \mathcal{A})(Ef) = EE^* d(Ef) = E(d + E^* dE)f \implies \mathcal{A} = E^* dE \tag{22}$$

The resulting connection is $SU(n)$ connection indeed:

$$\mathcal{A}^* = dE^* E = -E^* dE = -\mathcal{A} \tag{23}$$

The curvature $F$ of $\mathcal{A}$ is

$$F = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = dE^* \wedge dE + E^* dE \wedge E^* dE \tag{24}$$

It's not hard to see that $F$ has type $(1, 1)$. First, we rewrite (24) in terms of the orthogonal projector

$$F = -E^* d(1 - EE^*) \wedge dE = -E^* d(1 - P) \wedge dE \tag{25}$$

By construction of the bundle $E$ we have

$$(1 - P)dE = A(z)X + A(\sigma z)Y \tag{26}$$

for some matrix-valued 1-forms $X, Y$. Now we use (26) and the orthogonality conditions: $A(z)$, $A(\sigma z)$, $E$ are all pairwise orthogonal.

$$A^*(z)A(z)X = A^*(z)dE = -dA^*(z)E \implies X \text{ is } (0, 1) \text{ form} \tag{27}$$

$$A^*(\sigma z)A(\sigma z)Y = A^*(\sigma z)dE = -dA^*(\sigma z)E \implies Y \text{ is } (1, 0) \text{ form} \tag{28}$$

Using again the orthogonality of $E$ to $A(z)$, $A(\sigma z)$, we deduce from (25), (26) that $F$ is of type $(1, 1)$.

Remember that $\mathcal{A}$ is a connection in $E$ and what we are primarily interested in is the corresponding connection $\hat{\mathcal{A}}$ in $\hat{E}$. To extract some information about $\hat{\mathcal{A}}$ we return to the map $\pi : \mathbb{C}P^3 \to S^4$ and consider its restriction to an open set:

$$\begin{aligned} \pi(z_1 : z_2 : z_3 : z_4) &= (z_1 + z_2 j : z_3 + z_4 j) \sim \\ &\sim \Big( \frac{\bar{z}_3 z_1 + z_4 \bar{z}_2 + (-z_4 \bar{z}_1 + \bar{z}_3 z_2)j}{|z_3|^2 + |z_4|^2} : 1 \Big) \end{aligned} \tag{29}$$

At this point $z_1$, $z_2$ can be considered as local complex coordinates on $\mathbb{R}^4$, while $z_3$, $z_4$ parameterize complex structures, cf. (18). Therefore if $\pi^* \hat{F}$ has type $(1, 1)$ then $\hat{F}$ has type $(1, 1)$ for all compatible complex structures, so by (20) $\hat{F}$ is anti-self-dual.

# 6 ADHM equations

It remains to derive the equations on the matrix $A$ that was used in the construction of the bundle.

$$A = z_1 A_1 + z_2 A_2 + z_3 A_3 + z_4 A_4 \tag{30}$$

$A^*(\sigma z)A(z) = 0$ is equivalent to

$$\begin{aligned} A_1^\dagger A_1 - A_2^\dagger A_2 &= 0 \\ A_3^\dagger A_3 - A_4^\dagger A_4 &= 0 \\ A_1^\dagger A_3 - A_4^\dagger A_2 &= 0 \\ A_1^\dagger A_4 + A_3^\dagger A_2 &= 0 \\ A_1^\dagger A_2 &= 0 \\ A_3^\dagger A_4 &= 0 \end{aligned} \tag{31}$$

Bases changes in $W \simeq \mathbb{C}^k$, $V \simeq \mathbb{C}^{2k+n}$ induce transformations

$$A_i \to U A_i L,\ U \in U(2k+n),\ L \in GL(k) \tag{32}$$

By $A_{3,4} \to A_{3,4}L$ we simultaneously diagonalize $A_3^\dagger A_3$ and $A_4^\dagger A_4$.

Then we use $A_{3,4} \to UA_{3,4}$ to put the matrices into the block-diagonal form (it is possible due to $A_3^\dagger A_4 = 0$)

$$A_3 = \begin{bmatrix} D_1 \\ 0_{k\times k} \\ 0_{n\times k} \end{bmatrix},\ A_4 = \begin{bmatrix} 0_{k\times k} \\ D_2 \\ 0_{n\times k} \end{bmatrix} \tag{33}$$

Then we use $L$ again to make $D_1 = Id$. The equations on $A_3, A_4$ reduce to $D_2^\dagger D_2 = Id$. So apply $U$ again to set $D_2 = -Id$:

$$A_3 = \begin{bmatrix} Id_{k\times k} \\ 0_{k\times k} \\ 0_{n\times k} \end{bmatrix},\ A_4 = \begin{bmatrix} 0_{k\times k} \\ -Id_{k\times k} \\ 0_{n\times k} \end{bmatrix} \tag{34}$$

Then we use

$$\begin{aligned} A_1^\dagger A_3 - A_4^\dagger A_2 &= 0 \\ A_1^\dagger A_4 + A_3^\dagger A_2 &= 0 \end{aligned} \tag{35}$$

to conclude

$$A_1 = \begin{bmatrix} B_1^\dagger \\ B_2^\dagger \\ I^\dagger \end{bmatrix}, \; A_2 = \begin{bmatrix} B_2 \\ -B_1 \\ J \end{bmatrix} \tag{36}$$

The two remaining equations

$$\begin{aligned} A_1^\dagger A_2 &= 0 \\ A_1^\dagger A_1 - A_2^\dagger A_2 &= 0 \end{aligned} \tag{37}$$

acquire the form (ADHM equations)

$$\begin{aligned} [B_1, B_2] + IJ &= 0 \\ [B_1^\dagger, B_1] + [B_2^\dagger, B_2] - II^\dagger + J^\dagger J &= 0 \end{aligned} \tag{38}$$

# 7 Conclusion

We provided a self-consistent review of ADHM construction. The one point that can be added concerns the proof of "$F$ has type $(1,1)$". We preferred to provide a direct computation although the fact has simple geometric meaning. Namely, if we are given a bundle with holomorphic and unitary structures (which, of course, are given by different sets of local trivializations) and a connection that is compatible with both of them, then the curvature of this connection has type $(1,1)$. The connection $\mathcal{A}$ is by construction compatible with both structures, so the calculation can actually be omitted.

# References


[1] M. F. Atiyah, *Geometry of Yang-Mills Fields*, Edizioni della Normale, 1979