Donaldson-Thomas invariants

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1 Introduction

My talk aimed at defining Donaldson-Thomas invariants - the main outcome of the Donaldson-Thomas theory, and at discussing essential ingredients the definition requires, as was first explained in [Tho01]. The central idea of the Donaldson-Thomas theory is to equip Hilbert scheme $\mathrm{Hilb}(X)$ of subschemes in a projective Calabi-Yau threefold X with a symmetric obstruction theory. For this one has to study deformation theory of sheaves on X. Once the symmetric obstruction theory on $\mathrm{Hilb}(X)$ is introduced, one defines virtual structure sheaf of homological degree 0 on $\mathrm{Hilb}(X)$. DT invariant of X is then defined as degree of the corresponding cycle on $\mathrm{Hilb}(X)$.

In what follows I won't specify parameters for $\mathrm{Hilb}(X)$ throughout general discussion. Any of them will work equally well, except one has to consider subschemes of codimenshion at least 2.

1.1 Plan

Plan for the main part of my talk was the following.

(1) Reminder on Ext-groups;

(2) Deformation theory for sheaves;

(3) Virtual structure sheaves;

(4) Numerical DT-invariants and some examples.

2 Main Part

2.1 Ext - functors

This was a reminder on Ext - functors. These are defined as right derived functors of (global) Hom. I reminded main properties and discussed Yoneda's interpretation of Ext^1. The material is classical. A possible reference could be [aut], tags 010I and 06XP.

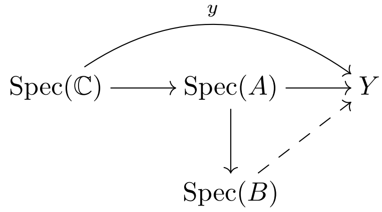
As an example, I described a calculation of Ext's between line bundles on $\mathbb{P}^1$. It involved Serre duality theorem and relation between Ext's and coherent cohomology.

2.2 Deformation theory for sheaves

This was an attempt to give example-based overview of the corresponding part of [Tho01].

First, I recalled general ideas and facts deformation theory exploits. Among them were lemmas 2.1 and 2.2, both stated without proofs.

Lemma 2.1. *Let Y be a scheme and $y\colon \mathrm{Spec}(\mathbb{C}) \to Y$ be it's point. Then Y is smooth at y iff for every pair (A, B) of local Artinian algebras and every surjective map $B \to A$ with kernel J, such that $J^2 = 0$, every solid diagam of the form*



can be completed by dotted arrow.

Lemma 2.2. *Take a scheme Y. Suppose it can be embedded into a bigger smooth scheme $\mathcal{Y}$:*

$$Y \hookrightarrow \mathcal{Y}.$$

Denote by $\mathcal{I}$ the ideal defined by this embedding. There is complex

$$NL_Y = \qquad (\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega^1_{\mathcal{Y}}|_Y)$$

$$-1 \qquad\qquad 0$$

defined by this embedding. Then obstructions for Y lie in $\mathcal{E}xt^1_{\mathcal{O}_Y}(NL_Y, \mathcal{O}_Y)$.

My core example to illustrate lemmas 2.1 and 2.2, as well as concepts of deformations and obstructions, was a scheme

$$\mathrm{Spec}(\frac{\mathbb{C}[x,y]}{(xy)})$$

sometimes called coordinate cross.

Along the way I introduced lemmas 2.3 and 2.4 (both are from [Tho01]) - the cornerstones of this section.

Lemma 2.3. *Take a point $x\colon \mathrm{Spec}(\mathbb{C}) \to \mathrm{Hilb}(X)$, which is given by an ideal sheaf $\mathcal{E}_0 \in \mathsf{Coh}(X)$. Then the tangent space $T_x \mathrm{Hilb}(X)$ is isomorphic to $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}_0, \mathcal{E}_0)$.*

In other words, deformations of $\mathcal{E}_0$ to an ideal sheaf over $\mathrm{Hilb}(X) \times \mathrm{Spec}(\mathbb{C}[t]/t^2)$, flat over $\mathrm{Spec}(\mathbb{C}[t]/t^2)$, are in 1-to-1 with $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}_0, \mathcal{E}_0)$.

Lemma 2.4. *The obstruction space at the point x is equal to $\mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{E}_0, \mathcal{E}_0)$*

A proof for the lemma 2.3 was provided. The lemma 2.4 was stated without proof.

2.3 Virtual structure sheaves

In this section I stated without proof theorem 2.5 - one of the main results of [Tho01].

Theorem 2.5. *Tangent-obstruction theory of $\mathrm{Hilb}(X)$ given by Ext^1 and Ext^2 is perfect. I.e. there is a complex of vector bundles*

$$P = \qquad E_0 \longrightarrow E_1$$

$$0 \qquad\quad 1$$

on $\mathrm{Hilb}(X)$ such that $H^0(P)$ is tangent sheaf and $H^1(P)$ is the obsturction sheaf.

The notion of a perfect tangent-obstruction theory was exemplified as follows.

- Suppose we are given a scheme Y embedded into a smooth bigger scheme $\mathcal{Y}$. Suppose further that Y is cut out inside $\mathcal{Y}$ by a section $s \in H^0(\mathcal{Y}, E)$ of a vector bundle on $\mathcal{Y}$. Then a morphism of complexes represented by the diagram

$$\begin{array}{rccc} & E^\vee|_Y & \longrightarrow & \Omega^1_{\mathcal{Y}}|_Y \\ & \downarrow{\scriptstyle s^\vee} & & \downarrow{\scriptstyle id} \\ NL_Y = & \mathcal{J}/\mathcal{J}^2 & \longrightarrow & \Omega^1_{\mathcal{Y}}|_Y \end{array}$$

corresponds to a perfect tangent-obstruction theory on Y.

Then I tried to elaborate on the following definition which is central to the Donaldson-Thomas theory.

Definition 2.6. In the situation described just above one can define a virtual structure sheaf $\mathcal{O}^{vir}_Y$ on Y as depicted below. Consider the Koszul complex

$$0 \to \wedge^{top}_{\mathcal{O}_\mathcal{Y}} E^\vee \to \ldots \to \wedge^1_{\mathcal{O}_\mathcal{Y}} E^\vee \to \mathcal{O}_\mathcal{Y} \to 0. \tag{2.7}$$

Then set

$$\mathcal{O}^{vir}_Y := \sum_{n \geq 0} (-1)^n [\wedge^n E^\vee] \in K(Y).$$

◇

As a motivation for the definition I gave a claim that $\mathcal{O}_Y^{vir}$ is just a K-theoretical counterpart for the (properly defined, see [Ful98]) intersection class of $s(\mathcal{Y})$ with $s_E(\mathcal{Y})$ in $\mathrm{Tot}(E)$, where $s_E\colon \mathcal{Y} \to \mathrm{Tot}(E)$ is the zero section. (This is not hard to see) Among the advantages of $\mathcal{O}_Y^{vir}$ is (propely understood) deformation invariance which in turn implies deformation invariance for DT invariants. I concluded section mentioning the fact that a virtual structure sheaf can be defined for any scheme, provided it is equipped with a perfect tangent-obstruction theory, but it requires a bit more advanced technique.

2.4 Numerical DT invariants and some examples

In this section I moved on to defining DT invariants. First, I stated without proof the theorem 2.8 for which [CK07] is a possible reference.

Theorem 2.8. *Given a scheme Y equipped with a perfect tangent-obstruction theory*

$$P = \qquad E_0 \longrightarrow E_1$$

$$0 \qquad\qquad 1 \qquad\qquad ,$$

the homology class of the corresponding virtual structure sheaf has dimenshion $rk(E_0) - rk(E_1)$.

After that I remarked that the homology class corresponding to the virtual structure sheaf of the tangent-obstruction theory from 2.5 has dimenshion zero. This follows from Serre duality and allows to define DT invariant to be equal to degree this class.

3 Conclusion

To conclude my talk I did a simple calculation of virtual number of points on a quintic threefold.

References

[Ful98] William Fulton. *Intersection Theory.* Springer, 1998.

[Tho01] Richard Thomas. "A holomorphic Casson invariant for Calabi-Yau 3-folds, and bundles on *K3* fibrations". In: *http://arxiv.org/abs/math/9806111v4* (2001).

[CK07] Ionuţ Ciocan-Fontanine and Mikhail Kapranov. "Virtual fundamental classes via dg-manifolds". In: *http://arxiv.org/abs/math/0703214v2* (2007).

[aut] The stacks project authors. "The Stacks Project". In: *https://stacks.math.columbia.edu/* ().