Quantum Hamiltonian reducton and quantum Calogero-Moser integrable system

1 Introduction

This report is based on a talk given during the Second Spring Student School on Mathematical and Physics. The topic of this talk was an introduction to the notion of quantum Hamiltonian reduction and usage of the latter as a tool in construction of quantum integrable systems. The talk was mostly based on lecture notes by Pavel Etingof [1].

In Section 2 we remind the notion of classical Hamiltonian reduction and give it algebraic reformulation more suitable for our quantum needs. In Section 3 we introduce main objects of our talk such as quantization of a Poisson algebra, quantum moment map and Hamiltonian reduction. In Section 4 we give nontrivial example of quantum Hamiltonian reduction and argue that it indeed may be thought of as a quantization of classical Calogero-Mozer integrable system

2 Classical Hamiltonian reduction

2.1 Geometric reminder

Let us briefly recall classical picture. Let (M, ω) be a symplectic manifold, $A = C^\infty(M)$ — algebra of continuous functions on M. A has a structure of Poisson algebra given by bracket $\{f, g\} := \omega^{-1}(df, dg)$. Map $ad : f \mapsto \{f, .\}$ gives a Lie algebra homomorphis from A to symplectic vector fields $\mathrm{Vect}_\omega(M) = \mathrm{Der}_{\{,\}}(A)$. The fields in the image of this map are called Hamiltonian fields.

Let G be a Lie group and G act on M via symplectomorphisms. Then differential of this action is a Lie algebra homomorphism from $\mathfrak{g}$ to $\mathrm{Vect}_\omega(M)$ $X : \mathfrak{g} \to \mathrm{Vect}_\omega(M)$. The following definition answers the question of whether obtained fields are Hamiltonian.

Definition 2.1. *(Classical) moment map is a homomorphis of Poisson algebras* $\rho : S\mathfrak{g} \to A$*, such that* $\forall \xi \in \mathfrak{g}, u \in A$

$$\{\rho(\xi), u\} = X_\xi u.$$

Note that usually this definition is given in terms of dual map $\mu = \rho^* : M \to \mathfrak{g}^*$.
Now let $\mathcal{O} \subset \mathfrak{g}^*$ be a coadjoint orbit.

Definition 2.2. *Hamiltonian reduction along the coadjoint orbit is a manifold*

$$M//_{\mathcal{O}}G := \rho^{*-1}(\mathcal{O})/G$$

2.2 Algebraic approach

Let us recast last definition on dual algebraic language. $S\mathfrak{g}$ can be thought of as an algebra of functions of $\mathfrak{g}^*$. Coadjoint orbit $\mathcal{O}$ corresponds to an ideal $I_{\mathcal{O}} = \{x \in S\mathfrak{g} \,|\forall \alpha \in \mathcal{O}, x(\alpha) = 0\}$.

Proposition 2.3. *Let* $J = \rho(I_{\mathcal{O}})A = \{v \in A|v = \sum \rho(x_i)u_i, x_i \in I_{\mathcal{O}}, u_i \in A\}$*, then*

$$A//\,\mathfrak{g} := C^\infty(M//_{\mathcal{O}}G) = (A/J)^{\mathfrak{g}}$$

This can be easily seen from the following diagrams.
Manifolds: $M \hookleftarrow \rho^{*-1}(\mathcal{O}) \twoheadrightarrow (\rho^{*-1}(\mathcal{O}))/G$
Algebra of functions: $A \twoheadrightarrow A/\rho(I_{\mathcal{O}})A \hookleftarrow (A/\rho(I_{\mathcal{O}})A)^{\mathfrak{g}}$
Note, that A/J in general is not Poisson algebra just as $\rho^{*-1}(\mathcal{O})$ is not symplectic manifold.

Proposition 2.4. $(A/J)^{\mathfrak{g}}$ *is a Poisson algebra.*

Proof. Let $v \in A$ be an element of some $\mathfrak{g}$–invariant equivalence class $\tilde{v} \in A/J$. $\mathfrak{g}$–invariance of $\tilde{v}$ implies that $\forall \xi \in \mathfrak{g}$

$$\{\rho(\xi), v\} \in J$$

For bracket on $(A/J)^{\mathfrak{g}}$ to well–defined it is sufficient to prove that $\forall w \in J$

$$\{v, w\} \in J$$

$\forall w \in J = \rho(I_{\mathcal{O}})A$ there are some $a_i \in I_{\mathcal{O}}, u_i \in A$ such that $w = \sum \rho(a_i)u_i$.

$$\{v, \sum \rho(\xi_i)u_i\} = \sum \{v, \rho(\xi_i)\}u_i + \sum \rho(\xi_i)\{v, u_i\} \in J$$

Where the first summand lives in J due to $\mathfrak{g}$–invariance. □

3 Quantum Hamiltonian reduction

3.1 Quantization of a Poisson algebra

Let A_0 be Poisson algebra and $A = A_0[\hbar]$ — vector space of polynomials in $\hbar$ with coefficients in A_0.

Definition 3.1. *A is called quantiztion of A_0 if it is equipped with an associative unital multiplication such that $\forall \hat{a}, \hat{b} \in A$, $\forall a, b \in A_0$, $\hat{a} = a + O(\hbar)$, $\hat{b} = b + O(\hbar)$*

$$\frac{1}{\hbar}[\hat{a}, \hat{b}] = \{a, b\} \ mod \ \hbar$$

Remark: As we would see from examples, it often useful to set $\hbar$ equal to some nonzero complex number $c \in \mathbb{C}^*$. The choice of c is irrelevant — different c's usually give isomorphic algebras.

Example 3.2. $A_0 = C^\infty(T^*\mathbb{R})$ *— is a Poisson algebra generated by x, p with a bracket $\{p, x\} = 1$. Than the following algebra is quantization of A_0*

$$A = \left\langle \hat{x}, \hat{p}, \hbar \;\middle|\; [\hat{p}, \hat{x}] = \hbar, \ [\hbar, \hat{x}] = [\hbar, \hat{p}] = 0 \right\rangle$$

Let $c \in \mathbb{C}^*$. Than $A_c := A/(\hbar - c) = D(\mathbb{R})$

This example can easily generalized to other manifolds — $D(N)$ — algebra of differential operators on a manifold N — is a quantization of $C^\infty(T^*N)$.

Example 3.3. *Recall that $A_0 = S\mathfrak{g}$ has a structure of a Poisson algebra given by Constant-Kirillov bracket. It admits the following quantization.*

$$A = T\mathfrak{g}[\hbar]/x \otimes y - y \otimes x - \hbar[x, y]$$

As usual, for all $c \in \mathbb{C}^*$, $A_c := A/(\hbar - c) = U(\mathfrak{g})$

3.2 Quantum Hamiltonian reduction

Now we can define quantum Hamiltonian reduction. All definitions can be immediately obtained from the classical ones buy replacing it's ingredients with their quantum versions.

Let A be an associative unital algebra equipped with an action of a group $\Phi \colon G \to Aut(A)$. This gives a map $X := d_e\Phi \colon \mathfrak{g} \to Der(A)$, where $Der(A) = \{j : A \to A | j(uv) = j(u)v + uj(v)\}$ is an algebra of derivations of A. One can ask himself whether the image of X lies in inner derivations.

Definition 3.4. *The homomorphism (of associative algebras) $\rho : U(\mathfrak{g}) \to A$ is a quantum moment map if $\forall \xi \in \mathfrak{g}, u \in A$*

$$[\rho(\xi), u] = X(\xi)u$$

Example 3.5. *Let $A = D(N)$ and G acts on N. Differential of this map $X : \mathfrak{g} \to Vect(N) \subset D(N)$ is a quantum already a moment map.*

Now we are finally ready to give our main definition.

Definition 3.6. *Let $J := \rho(\xi)A, \xi \in \mathfrak{g}$. The following algebra is called quantum Hamiltonian reduction*

$$A/\!/\,\mathfrak{g} := (A/J)^{\mathfrak{g}}$$

Note that J in general is not a two-sided ideal. Thus it's not automatically obvious that $A/\!/\,\mathfrak{g}$ is an associative algebra. The next proposition is quantum version of Proposition 2.4.

Proposition 3.7. *$A/\!/\,\mathfrak{g}$ is an associative algebra*

Proof. The proof is analogous to the proof of Proposition 2.4. Let u, v be an elements of $\mathfrak{g}$–invariant classes in A/J, i.e. $\forall \xi \in \mathfrak{g}$

$$[\rho(\xi), u] \in J$$

For $(A/J)^{\mathfrak{g}}$ to be an algebra it is sufficient to prove that $\forall u_i, v_i \in A, \xi_i, \eta_i \in \mathfrak{g}$

$$(u + \sum \rho(\xi_i)u_i)(v + \sum \rho(\eta_j)v_j) - uv \in J$$

By trivial rearrangements we arrive to an expression where every term is an element of J

$$\sum_i \rho(\xi_i)u_i(v + \sum \rho(\eta_j)v_j) + \sum \rho(\eta_j)uv_j + \sum [u, \rho(\eta_j)]v_j \in J$$

□

In classical case we had more general definition depending on a coadjoint orbit $\mathcal{O}$. This hints us the following definition

Definition 3.8. *Let $I \subset U(\mathfrak{g})$ be a two-sided ideal. Then*

$$A/\!/_{\mathcal{O}}\,\mathfrak{g} := (A/\rho(I)A)^{\mathfrak{g}}$$

Remark: The condition of I being two-sided is required for proper action of $\mathfrak{g}$ on $A/\rho(I)A$. Indeed, we want that $\forall \xi \in \mathfrak{g}, \eta_i \in I, v_i \in A$ the following holds.

$$[\rho(\xi), \sum_i \rho(\eta_i)v_i] = \sum_i ([\rho(\xi), \rho(\eta_i)]v_i + \rho(\eta_i)[\rho(\xi), v_i]) =$$

$$\sum_i \rho([\xi, \eta_i])v_i = 0 \pmod{J}$$

Let us proceed with some examples.

Example 3.9. *Let N be a manifold and group G acts on it freely. This action gives rise to an action of $\mathfrak{g}$ on $D(N)$. Than*

$$D(N)//\,\mathfrak{g} = D(N/G)$$

We won't go into the details of the proof. One can reproduce it by adapting the proof of classical statement $T^*N//G = T^*(N/G)$.
Let's study another example, which would be useful for us in the future. Namely reduction vector space with respect to
Let $V = \mathbb{R}^n$ and $G = \mathbb{R}$ acts on V via dilatations, i.e. $a \in G : v \mapsto av$. Let fix in $\mathfrak{g}$ some basis element ξ. Consider classical case first.

- The manifold we want to reduce is $T^*V = V \times V^* = \{(p, q) | p \in V^*, q \in V\}$.
- The action of G is $a \in G : (p, q) \mapsto (\frac{1}{a}p, aq)$
- The moment map is given by $\rho_0(\xi) = p(q) = \sum p_i q_i$.
- Finally, we need to choose the coadjoint orbit. G is abelian, thus every coadjoint orbit is a single point. We can choose arbitary $\mathcal{O}_\lambda = \{\lambda\xi^*\}$
- The ideal corresponding to $\mathcal{O}$ is $I_{\mathcal{O}} = (\xi - \lambda)\, S\mathfrak{g}$
- Finally, the reduction is $T^*V//_\lambda\mathbb{R}^* = \{(p, q) | p(q) = \lambda\}/(p, q) \sim (a^{-1}p, aq)$. For $\lambda = 0$ this is $T^*\mathbb{R}P^{n-1}$.

Now we take a look at the quantum version.

- We start from differential operators on V. Explicitly $A = D(V) = \langle q_i, \frac{\partial}{\partial q_i} | [q_i, \partial_j] = -\delta_{ij}, [q_i, q_j] = [\partial_i, \partial_j] = 0 \rangle$
- The moment map is given by Euler vector field $\rho(\xi) = \sum q_i \partial_i$

- Since $U(\mathfrak{g})$ is isomorphic to $S\mathfrak{g}$ we choose quantum ideal to be the same as classical $I_{\mathcal{O}_\lambda} = (\xi - \lambda)U(\mathfrak{g})$. Image of this ideal in $D(V)$ is than $J = (q^i\partial_i - \lambda)A$.
- Now we argue that quantum reduction is $A//_\lambda\,\mathfrak{g} = \langle q_i, \partial_i | q_i\partial_i = \lambda\rangle^{\mathfrak{g}}$. Indeed:
  - For $r \in A$ $\mathbb{R}$–invariance implies that $deg(r) = 0$
  - Taking quotient along J (+homogeneity) means $q^i\partial_i = \lambda$
- Algebra $A//_\lambda\,\mathfrak{g}$ has a nice representation τ on the space $W_{\frac{\lambda}{n}} = \{(q_1 \dots q_n)^{\frac{\lambda}{n}} f(q_1, \dots, q_n), f \in \mathbb{C}[q_1^{\pm 1}, \dots]|\text{deg} f = 0\}$.

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3.3 Dictionary

Finally we give a brief summary of previous chapters in form of table that compares same objects in different settings.

Geometry	**Algebra**	**Quantum Algebra**
M	$A_0 = C^\infty(M)$	A
$G \to Diff_\omega(M)$	$\mathfrak{g} \to Der_{\{,\}}(A_0)$	$\mathfrak{g} \to Der(A)$
$\mu : M \to \mathfrak{g}^*$	$\rho_0 : S\mathfrak{g} \to A_0$	$\rho : U(\mathfrak{g}) \to A$
$\mathcal{O} \subset \mathfrak{g}^*$	$I_{\mathcal{O}} = \ker(S\mathfrak{g} \twoheadrightarrow \mathbb{R}[\mathcal{O}])$	$I = \ker(U(\mathfrak{g}) \twoheadrightarrow ?)$
$\mu^{-1}(\mathcal{O})$	$A_0/\rho_0(I_{\mathcal{O}})A$	$A/\rho(I)A$
$\mu^{-1}(\mathcal{O})/G$	$(A_0/\rho_0(I_{\mathcal{O}})A)^{\mathfrak{g}}$	$(A/\rho(I)A)^{\mathfrak{g}}$
$T^*N//_0 G = T^*(N/G)$	$A_0//_0\,\mathfrak{g} = C^\infty(T^*N/G)$	$A//_0\,\mathfrak{g} = D(N/G)$

4 Quantum Calogero-Mozer system

4.1 Quantum CM orbit

To construct a quantum reduction we need three ingredients

1. Associative algebra A
2. Action of a Lie algebra $\mathfrak{g}$
3. Two-sided ideal I in $U(\mathfrak{g})$

From what we know of classical Calogero-Mozer space the first two ingredients are obvious — $A = D(\mathfrak{gl}_n)$ and $\mathfrak{g} = \mathfrak{sl}_n$ and the action is lifted to A from the adjoint action. Now let us motivate the choice of ideal.
As usual, we start with classical coadjoint orbit.

Proposition 4.1. *Let $\mathfrak{g} = \mathfrak{sl}_n$ and $\mathcal{O}_C \subset \mathfrak{g}^*$ is a coadjoint orbit of the following matrix $C = diag(n-1, -1 \ldots, -1)$. Then*

$$\mathcal{O}_C = \{(p, q) \in V^* \times V | p(q) = n\}/(p, q) \sim (a^{-1}p, aq)$$

Proof. Note that the second space is the reduction from our example from section 3. The explicit diffeomorphism $\psi : T^*V//_n\mathbb{R} \to \mathcal{O}_C$ is

$$\psi : (p, q) \mapsto M = q \otimes p - \mathbb{E}$$

It's easy to see that matrix is conjugated to C if and only if it is of the form $q \otimes p - \mathbb{E}$ and $p(q) = n$. Also condition $(p, q) \sim (a^{-1}p, aq)$ ensures that map is injective. □

The corresponding ideal in $S\mathfrak{g}$ is $I_{\mathcal{O}_C} = \ker(\sigma_0 : S\mathfrak{g} \twoheadrightarrow \mathbb{R}[\mathcal{O}_C])$. We can recover σ_0 from the action of $\mathfrak{sl}_n$ on $\mathcal{O}_C$. With some laziness in notations this action reads

$$ad_{E_{ij}} M = [E_{ij}, M] = \{q_i p_j, M\}$$

Thus we arrive to $\sigma_0 : E_{ij} \mapsto q_i p_j$ (what about central extension? It actually now seems to me that the right answer is $q_i p_j - \delta_{ij}$)
The quantization of σ_0 is a map $\sigma_k : U(\mathfrak{g}) \twoheadrightarrow D(V)//_{kn}\mathbb{R}$.

$$\sigma(E_{ij}) = q_i \partial_j$$

Remark: Note that classical consideration suggests that we should only care for $k = 1$. We would like to consider generic k since it would lead us to different coupling constants for CM integrable system. If one chooses $k \neq 1$ in constuction of classical CM he would get equivalent integrable system.

The quantum ideal is then $I_k = \ker(\sigma_k)$. Recall that $D(V)//_{kn}\mathbb{R}$ has a faithful representation τ. This allows us to think of $I_k = \ker(\phi := \tau\sigma_k)$ as annihilator of a representation $\mathfrak{sl}_n$ (W_k, ϕ).

4.2 Quantum CM space

Let us stress it again — $I_k \subset U(\mathfrak{g})$ is an annihilator of representation (W_k, ϕ)

- $W_k = \{(q_1 \dots q_n)^k f(q_1, \dots, q_n), f \in \mathbb{C}[q_1^{\pm}, \dots] | \deg f = 0\}$
- $\phi(E_{ij}) = q_i \partial_j, \;\; i \neq j$
- $\phi(E_{ii} - E_{jj}) = q_i \partial_i - q_j \partial_j$

Definition 4.2. $B_k := D(\mathfrak{gl}_n)//_{I_k} \mathfrak{sl}_n$ *is called quantum Calogero-Mozer space.*

B_k has a natural action on the space of equivariant functions on $\mathfrak{gl}_n^{reg}$ — the space of $n \times n$ matrix with different eigenvalues — with values in W_k

$$E_k = (C^\infty(\mathfrak{gl}_n^{reg}) \otimes W_k)^{\mathfrak{g}}$$

To understand this action better we give the following proposition.

Proposition 4.3. *Space E_k can be identified with functions $C^\infty(\mathfrak{h}^{reg})$*

Proof. Let $F \in E_k$. $\mathfrak{g}$–invariance implies that $\forall x \in \mathfrak{sl}_n, y \in \mathfrak{gl}_n^{reg}$

$$\phi(e^x) F(e^{-x} y e^x) = F(y)$$

First, we can choose x so that $e^{-x} y e^x$ is diagonal. This means that the function F is defined by it's values on $\mathfrak{h}$.

Next we can choose $y \in \mathfrak{h}^{reg}, x \in h$. This gives us

$$F(y) = \phi(e^x) F(y)$$

This means that for $y \in \mathfrak{h}^{reg}$ $F(y)$ is in $W_k[0]$, the zero-weight subspace of W_k. But it's easy to see that $W_k[0] = \mathbb{C}(q_1 \dots q_n)^k$ is one-dimensional. □

This identification allows us to construct a map $\pi_k : D(\mathfrak{g})^g \to D(\mathfrak{h})$.

4.3 Quantum CM system

To construct an integrable system via quantum Hamiltonian reduction we need a family of commuting $\mathfrak{g}$–invariant operators in the base algebra A. If the size of this family is "big enough" comparing to the "size" of reduction we would get integrable system. One can look up precise definition in Sectoin 5.3 of [1]. In our case we would give interpretation of our integrable system

as a family of n differential operators on n-dimensional space which definitely suffice.
The family of n independent $\mathfrak{sl}_n$–invariant differential operators on $\mathfrak{gl}_n$ is $\{H_r = Tr\left(\left(\frac{\partial}{\partial E_{ij}}\right)^r\right) | r = 1..n\}$.

Theorem 4.4. *Let $\{x_1, \dots x_n\}$ be a set of standard coordinates on $\mathfrak{h}$, $L'_r = \pi_k(H_r)$, $\delta = \Pi_{\alpha>0}\alpha(x) = \Pi_{i<j}(x_i - x_j)$ and $L_r = \delta L'_r \delta^{-1}$.*

1. *Operators $\{L_r\}$ form an integrable system*
2. $$L_2 = \sum_{i=1}^{n} \partial_i^2 - \sum_{i \neq j} \frac{k(k+1)}{(x_i - x_j)^2}$$

Proof. The first part of the theorem is obvious. To prove the second part we need to recall how π_k was constructed.
Let $F \in E_k$. We need to calculate the action of

$$H_2 = \sum \partial_{x_i}^2 + 2\sum_{\alpha>0} \partial_{f_\alpha}\partial_{e_\alpha}$$

on $F(x)$ for $x \in \mathfrak{h}$. While the first part is already a nice operator on $\mathfrak{h}$, the second part moves us away from $\mathfrak{h}$. Let us calculate $(\partial_{f_\alpha}\partial_{e_\alpha}F)(x)$. $\mathfrak{g}$–invariance gives us

$$(\partial_{f_\alpha}\partial_{e_\alpha}F)(x) = \partial_t\partial_s|_{s=t=0}F(x + tf_\alpha + se_\alpha)$$

Note that

$$Ad(e^{s\frac{e_\alpha}{\alpha(x)}})(x + tf_\alpha + se_\alpha) = x + tf_\alpha + ts\alpha(x)^{-1}h_\alpha + \dots,$$

where we defined $h_\alpha := [e_\alpha, f_\alpha]$. This allows us to write

$$F(x + tf_\alpha + se_\alpha) = \phi(e^{-se_\alpha\alpha(x)^{-1}})F(x + tf_\alpha + ts\alpha(x)^{-1}h_\alpha + \dots) =$$
$$\phi(e^{-se_\alpha\alpha(x)^{-1}}e^{tf_\alpha\alpha(x)^{-1}})F(x + ts\alpha(x)^{-1}h_\alpha + \dots)$$

To proceed we need to compute $\phi(e_\alpha f_\alpha)|_{W_k[0]}$:

$$\phi(e_\alpha f_\alpha)(q_1 \dots q_n)^k = q_i\partial_j q_j\partial_i(q_1 \dots q_n)^k = k(k+1)(q_1 \dots q_n)^k$$

Substitution into Hamiltonian gives us

$$L'_2F(x) = \sum_{i=1}^{n} \partial_i^2 F(x) + 2\sum_{\alpha>0}(\alpha(x)^{-1}\partial_{h_\alpha} - k(k+1)\alpha(x)^{-2})F(x)$$

This already is a good result, but we would like to bring it to nicer form. In our standard basis $\alpha(x) = x_i - x_j$. Then

$$L_2' F(x) = \sum_{i=1}^{n} \partial_i^2 F(x) + 2 \sum_{\alpha>0} (\alpha(x)^{-1} \partial_{h_\alpha} F(x)) - \sum_{i>j} \frac{2k(k+1)}{(x_i - x_j)^2}$$

Note that

$$\delta^{-1} \sum \partial_i^2 \delta = \sum \partial_i^2 + 2 \sum_{\alpha>0} \alpha(x)^{-1} \partial_{h_\alpha}$$

This follows from the fact that $\sum \partial_i^2(\delta) = 0$.
Finally we arrive to a well-known Hamiltonian

$$L_2 = \sum_{i=1}^{n} \partial_i^2 - \sum_{i>j} \frac{2k(k+1)}{(x_i - x_j)^2}$$

□

Bibliography

[1] Pavel Etingof, Lectures on Calogero-Moser systems https://arxiv.org/abs/math/0606233