Thermodynamic Bethe ansatz and the Sine-Gordon model

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1 Introduction

The note and my talk are based on an article by A.B. Zamolodchikov paper [1]

Two dimensional Sine-Gordon model is given by the action

$$\mathcal{A}_{\mathrm{SG}_p} = \frac{1}{2}\int (\partial_a\varphi)^2 d^2x - 2\mu \int \cos\beta\varphi d^2x, \tag{1}$$

the $\varphi(x)$ is a real scalar field in $2d$ Euclidean space with coordinates $x^a = (x^0, x^1)$. Theories (1) depend on parameter $0 \leqslant \beta \leqslant \sqrt{8\pi}$, or, in more convenient parametrization

$$p = \frac{\beta^2}{8\pi - \beta^2}, \qquad 0 \leqslant p < \infty \tag{2}$$

Sine-Gordon model can be considered as a conformal free scalar field theory, perturbed by operators $\exp(\pm i\beta\varphi)$, which have a conformal dimension $\Delta_{\mathrm{SG}} = \beta^2/8\pi = p/(p+1)$. Then the free field correlation functions (at $\mu = 0$) are

$$\left\langle e^{i\beta\varphi(x_1)} \dots e^{i\beta\varphi(x_n)} e^{-i\beta\varphi(y_1)} \dots e^{-i\beta\varphi(y_n)} \right\rangle_{\mu=0} = \frac{\prod_{i>j}^n |x_i - x_j|^{4\Delta_{\mathrm{SG}}} |y_i - y_j|^{4\Delta_{\mathrm{SG}}}}{\prod_{i,j=1}^n |x_i - y_j|^{4\Delta_{\mathrm{SG}}}} \tag{3}$$

We consider a system of length L at temperature T (means that another length parameter is $1/T$). Ground state energy can be computed using the perturbation theory. In zeroth order in μ, the calculation repeats the Casimir effect for a free field between plates [2].

$$\mathcal{E}(L) \sim -\frac{\pi c_{\mathrm{UV}}}{6L} = -\frac{\pi}{6L}, \quad L \to 0. \tag{4}$$

In general, we can compute the energy, using conformal perturbation theory in powers of $\eta = L\mu^{\frac{p+1}{2}}$ in UV limit, namely

$$\mathcal{E}(L) = -\frac{\pi}{6L} k(L, \mu), \tag{5}$$

$$k(L, \mu) = \sum_{n=0}^{\infty} k_n \eta^{2n} \tag{6}$$

We also calculate ground state energy of the system with external current j_a using thermodynamic Bethe ansatz.

2 Main part

There is another point of view to UV limit. The problem with the approach described above (limit of small distances $L \to 0$) is the difficulty of applying the Bethe ansatz. We propose another limit with the presence of a strong external field A. The limit is $L \to \infty$, and $A \to \infty$. Action with a strong external field A is

$$\mathcal{A}\left[A_a\right] = \frac{1}{2}\int \left(\partial_a \varphi\right)^2 d^2x - 2\mu \int \cos\beta\varphi d^2x + \int j^a A_a d^2x, \tag{7}$$

the current is

$$j^a = \frac{\beta}{2\pi}\epsilon^{ab}\partial_b\varphi. \tag{8}$$

At UV limit $A \gg \mu^{(p+1)/2}$ we can apply perturbation theory. In zeroth and first order we obtain [1]:

$$\mathcal{E}(A,\mu) = -\frac{p}{\pi(p+1)}A^2 - \mu^2 A^{\frac{2(p-1)}{(p+1)}}\pi\left(\frac{2p}{p+1}\right)^{2(p-1)/(p+1)}\frac{\Gamma\left(\frac{1-p}{p+1}\right)}{\Gamma\left(\frac{2p}{p+1}\right)} \tag{9}$$

Let us briefly describe thermodynamic Bethe ansatz technique. Spectrum of the model (1) at $p > 1$ consists of topologically charged solitons (and antisolitons) $(s, \bar{s})$ of mass M. Two particle scattering amplitude is $S(\theta_1 - \theta_2) = \exp\left[i\delta_{ss}(\theta_1 - \theta_2)\right]$, where

$$\delta_{ss}(\theta_1 - \theta_2) = \int_0^\infty \frac{\sin\omega(\theta_1 - \theta_2)\sinh\frac{\pi\omega(p-1)}{2}}{\cosh\frac{\pi\omega}{2}\sinh\frac{\pi\omega p}{2}}\frac{d\omega}{\omega}. \tag{10}$$

Hamiltonian of the system in external field is

$$\mathcal{H}(A) = \mathcal{H}_{\mathrm{SG}_p} - AQ, \tag{11}$$

the Q is a soliton's charge. Momentum of particles is

$$p_0 = -A + M\cosh\theta, \quad p_1 = M\sinh\theta \tag{12}$$

The mass M of a particle on our picture is nothing but temperature $M = T$.

Thermodynamic Bethe ansatz equations are [3]

$$\frac{ML}{2\pi}\cosh\theta = \rho(\theta) - \int_{-B}^{B}\tilde{k}\left(\theta - \theta'\right)\rho\left(\theta'\right)d\theta', \quad \tilde{k}(\theta) = \frac{1}{2\pi}\frac{d\delta_{ss}(\theta)}{d\theta}. \tag{13}$$

The ρ is a density of energy levels. Note, that in ultraviolet limit equations are "linear". This is due to the uniform population of the energy levels [3]. Also, we see, that particles live in some "Fermi sphere"

$$-B \leq \theta \leq B. \tag{14}$$

Ground state energy is

$$\mathcal{E}(A) = -\frac{1}{L}\int_{-B}^{B}(A - M\cosh\theta)\rho(\theta)d\theta. \tag{15}$$

Solving these equations we obtain the resulf for ground state energy in UV limit (zeroth order)

$$\mathcal{E}_{\mathrm{UV}}(A) = -\frac{A}{2\pi}\epsilon(-\infty). \tag{16}$$

3 Conclusion

In this note, we have briefly described the computation of ground state energy for the Sine-Gordon model with a strong external field (ultraviolet regime). We did it in tho ways. Firstly, we computed the energy using the conformal perturbation theory with vertex operators $\exp(\pm i\beta\varphi)$. Secondly, we have used Bethe-ansatz equations (TBA) in the UV limit.In this case, they are simplified (linearization).

The results of both calculations are the same.

References

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