Plan

1. Calculating the number of collective coordinates of a space of K-instanton with $SU(N)$ using index theorem.
2. Self-dual equations in complex coordinates
3. The map $\gamma: \mathbb{CP}^3 \to S^4$ and self dual equations.

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1 Collective coordinates & index theorem.

1.1 Set up. We start with $SU(N)$ YM action:

$$S = -\frac{1}{2g^2}\int d^4x \, \mathrm{tr}(F_{\mu\nu}F_{\mu\nu}) \quad . \quad F_{\mu\nu} = F^a_{\mu\nu} T_a$$

$$[T_a, T_b] = f^c_{ab} T_c \ , \quad \mathrm{tr}(T_a T_b) = -\frac{1}{2}\delta_{ab}$$

Work with Euclidean signature. $\delta_{\mu\nu} = (+,+,+,+)$

Convention: $D_\mu Y = \partial_\mu Y + [A_\mu, Y]$ ← Lie algebra valued Y

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = [D_\mu, D_\nu] .$$

The classical eq. of motion : $D_\mu F_{\mu\nu} = 0$

The instanton number is defined as

$$\boxed{K = -\frac{1}{16\pi^2}\int d^4x \, F^a_{\mu\nu} * F^b_{\mu\nu} \underbrace{\mathrm{tr} T_a T_b}_{-\frac{1}{2}\delta_{ab}}}$$

$$*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma} \ ; \quad \epsilon_{1234} = 1 . \quad *^2 = 1 .$$

The action S can be rewritten as

$$S = -\frac{1}{2g^2}\int d^4x \, \mathrm{tr} F^2 = -\frac{1}{4g^2}\int d^4x \, (F \mp *F)^2 \mp \frac{1}{2g^2}\int d^4x \, \mathrm{tr} F * F \geq$$

$$\geq \mp \frac{1}{2g^2}\int d^4x \, \mathrm{tr} F_{\mu\nu} * F_{\mu\nu} = \frac{8\pi^2}{g^2}(\pm K) .$$

The solution to the field equations that minimizes the action has either a self-dual or antiselfdual field strength: $F_{\mu\nu} = \pm * F_{\mu\nu} \Rightarrow$

$$\boxed{\begin{cases} F_{12} = \pm F_{34} \\ F_{13} = \pm F_{42} \\ F_{14} = \pm F_{23} \end{cases}}$$

- Instantons (self dual $F_{\mu\nu}$) have $k>0$.
- Anti-instantons (anti self-dual $F_{\mu\nu}$) have $k<0$.

Our task is to find out the number of solutions to the self duality equations by relating it to the index of the Dirac operator.

For definiteness we consider anti-instons: $F_{\mu\nu} = -*F_{\mu\nu}$ and look for deformations which satisfy an antiself-duality equation.

1.2 From bosonic collective coordinates to the fermionic collective coordinates.

In this subsection we will relate the number of solutions o the antiself dual equation to the index of Dirac operator.

We will show that

bosonic collect. coord. = 2 # fermionic collective coord.

$F_{\mu\nu} = -*F_{\mu\nu}$, $A_\mu = A_\mu^{cl} + \delta A_\mu$,

$F_{\mu\nu}^{cl} = -*F_{\mu\nu}^{cl}$ with charge k.

we denote $\boxed{\phi_\mu = \delta A_\mu}$ & $\boxed{f_{\mu\nu} = D_\mu \phi_\nu - D_\nu \phi_\mu}$

interested in linear in ϕ_μ terms only:

$$D_\mu \phi_\nu - D_\nu \phi_\mu = -*(D_\mu \phi_\nu - D_\nu \phi_\mu) \quad \Big| \; D_\mu \text{ contains } A_\mu^{cl} \text{ only.}$$

$$f_{\mu\nu} = -*f_{\mu\nu} \implies \boxed{\bar{\sigma}_{\mu\nu} D_\mu \phi_\nu = 0}$$

where $\bar{\sigma}_{\mu\nu} = \frac{1}{2}(\bar{\sigma}_\mu \sigma_\nu - \bar{\sigma}_\nu \sigma_\mu)$; $\sigma_\mu = (\vec{\tau}, iI)$; $\bar{\sigma}_\mu = (\vec{\tau}, -iI)$

proof: $\mathrm{tr}(\bar{\sigma}_{\rho\sigma} \bar{\sigma}_{\mu\nu}) D_\mu \phi_\nu = \frac{1}{2} \mathrm{tr}([\bar{\sigma}_{\rho\sigma}, \bar{\sigma}_{\mu\nu}] + \{\bar{\sigma}_{\rho\sigma}, \bar{\sigma}_{\mu\nu}\}) D_\mu \phi_\nu =$

$$= D_\sigma \phi_\rho - D_\rho \phi_\sigma - \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma} f_{\mu\nu} = 0 .$$

We must also fix the gauge: all fluctuations must be orthogonal to gauge transformation.

The inner product $\langle \delta A_\mu , \delta' A_\mu \rangle \sim \int d^4x \, \mathrm{tr}(\delta A_\mu \delta' A_\mu)$.

gauge transformed field has the form $\delta A_\mu = D_\mu \Lambda$, so

$$\int d^4x \, \mathrm{tr}(D_\mu \Lambda \underbrace{\delta A_\mu}_{\phi_\mu}) = 0 \xrightarrow{\text{int. by parts}} \boxed{D_\mu \phi_\mu = 0}$$

$$\begin{cases} f_{\mu\nu} = - * f_{\mu\nu} \\ D_\mu \phi_\mu = 0 \end{cases} \Rightarrow \boxed{\bar\sigma_\mu \sigma_\nu D_\mu \phi_\nu = 0}$$

since $\bar\sigma_\mu \sigma_\nu = \delta_{\mu\nu} + \bar\sigma_{\mu\nu}$
„1 + 3"

Let us write them in terms of spinor indexes.

We introduce the following notation

$$\begin{cases} \bar{\not{D}} = \bar\sigma_{\mu,\alpha'\beta} D_\mu = \bar D_{\alpha'\beta} \\ \sigma_\nu^{\alpha'\beta} \phi_\nu = \Phi^{\alpha'\beta} \end{cases}$$

$$\boxed{\bar D \Phi = \bar D_{\alpha'\beta} \Phi^{\alpha'\gamma} = 0}$$

Note, that $\Phi^{\alpha'\beta}$ is in adjoint representation, since

$$\bar D_{\alpha'\beta} \Phi^{\alpha'\gamma} = \partial_{\alpha'\beta} \Phi^{\alpha'\gamma} + [A_{\alpha'\beta}, \Phi^{\beta\gamma}] = 0$$

We can represent the quaternion Φ as

$$\Phi = \begin{pmatrix} a & b^* \\ b & -a^* \end{pmatrix}$$

with a, b complex adjoint-valued function.

Then

$$\bar{\not{D}} \Phi = 0 \Rightarrow \boxed{\begin{cases} \bar{\not{D}} \lambda = 0 \\ \bar{\not{D}} \, i\sigma^2 \lambda^* = 0 \end{cases} , \quad \lambda = \begin{bmatrix} a \\ b \end{bmatrix}}$$

For each spinor solution $\lambda \to \underline{i\sigma^2 \lambda^*}$ is also a solution.

because $(\bar\sigma^\mu)^* = -\sigma^2 \bar\sigma^\mu \sigma^2$

Given a solution λ, we can construct two different solutions

$$\Phi^{(1)} = \begin{pmatrix} a & b^* \\ b & -a^* \end{pmatrix} ; \quad \Phi^{(2)} = \begin{pmatrix} ia & -ib^* \\ ib & ia^* \end{pmatrix}$$

The reason why we treat $\Phi^{(1)}$ and $\Phi^{(2)}$ as two independent solutions is a reality condition: δA^a_μ should be real.

Statement

$\Phi^{(1)}$ and $\Phi^{(2)}$ yield different variations for A^a_μ.

$$\left.\begin{aligned} \Phi^{(1)} &= \begin{pmatrix} \phi^a_3 + i\phi^a_4 & \phi^a_1 + i\phi^a_2 \\ \phi^a_1 - i\phi^a_2 & -\phi^a_3 + i\phi^a_4 \end{pmatrix} T_a \\ \Phi^{(2)} &= \begin{pmatrix} i\phi^a_3 - \phi^a_4 & -i\phi^a_1 + \phi^a_2 \\ i\phi^a_1 + \phi^a_2 & -i\phi^a_3 - \phi^a_4 \end{pmatrix} T_a \end{aligned}\right] \text{``}=\text{''} \begin{pmatrix} \delta A^a_3 + i\delta A^a_4 & \delta A^a_1 + i\delta A^a_2 \\ \delta A^a_1 - i\delta A^a_2 & -\delta A^a_3 + i\delta A^a_4 \end{pmatrix} T_a$$

which means that

$$\Phi^{(1)} : \delta A_1 = \phi_1 , \delta A_2 = \phi_2 ; \delta A_3 = \phi_3 , \delta A_4 = \phi_4$$

$$\Phi^{(2)} : \delta A_1 = -\phi_2 , \delta A_2 = \phi_1 , \delta A_3 = -\phi_4 , \delta A_4 = \phi_3$$

There are no further independent solution.

We get that $\not{D}\lambda = 0 \longrightarrow \Phi^{(1)}, \Phi^{(2)}$.

Therefore, the number of solutions for Φ is twice the number of solutions for a single two component adjoint spin

1.3 Fermionic moduli and index theorem.

In this subsection we study the Dirac equation

in the background of an anti-instanton,
i.e. # zero modes for massless four component Dirac fermion in an arbitrary representation (fundamental, adjoint) of an arbitrary gauge group.

We consider massless Dirac spinor

$$\gamma_\mu D_\mu \Psi = 0 \qquad \Psi = \begin{pmatrix} \lambda \\ \bar{\lambda} \end{pmatrix} \qquad \begin{cases} \lambda = \frac{1}{2}(1+\gamma^5)\Psi \\ \bar{\lambda} = \frac{1}{2}(1-\gamma^5)\Psi \end{cases}$$

$$\Rightarrow \begin{cases} \bar{\not{D}}\lambda = 0 \\ \not{D}\bar{\lambda} = 0 \end{cases}$$

Statement On the antiinstanton background $\bar{\lambda}$ has not zero modes.

proof. $\bar{\not{D}}\not{D} = D^2 + \frac{1}{2}\bar{\sigma}_{\mu\nu}F_{\mu\nu} = D^2$, since $\begin{cases} F_{\mu\nu} = -*F_{\mu\nu} \\ \bar{\sigma}_{\mu\nu} = *\bar{\sigma}_{\mu\nu} \end{cases}$

$$\not{D}\bar{\lambda} = 0 \Rightarrow D^2\bar{\lambda} = 0 \,\Big|\, \bar{\lambda}^* \,\&\, \int \;\Rightarrow\; \int |D_\mu\bar{\lambda}|^2 = 0$$

$$D_\mu\bar{\lambda} = 0 \Rightarrow F_{\mu\nu}\bar{\lambda} = 0 = F^a_{\mu\nu}T_a\bar{\lambda} \Rightarrow T_a\bar{\lambda} = 0$$

$$D_\mu = \partial_\mu + A^a_\mu T^a \Rightarrow D_\mu\bar{\lambda} = \partial_\mu\bar{\lambda} = 0 \Rightarrow \boxed{\bar{\lambda} = 0}$$

#

For the λ equation $\not{D}\bar{\not{D}} = D^2 + \frac{1}{2}\sigma_{\mu\nu}\overset{\neq 0}{F_{\mu\nu}}$.

- Knowing that $\not{D}$ has no zero modes, we conclude that

$$\boxed{\begin{aligned} &\ker\bar{\not{D}} = \ker\not{D}\bar{\not{D}} \\ &\ker\not{D} = \ker\bar{\not{D}}\not{D} = 0 \end{aligned}}$$

The index of the Dirac operator is defined as

$$\boxed{\operatorname{Ind}\not{D} = \dim\ker\not{D}\bar{\not{D}} - \dim\ker\bar{\not{D}}\not{D}.}$$

First of all,

Statement

$\not{D}\bar{\not{D}}$ and $\bar{\not{D}}\not{D}$ have the same non-zero eigenvalues.

proof: let ϕ_n be the orthogonal set of $\bar{\not{D}}\not{D}$:

$$\bar{\not{D}}\not{D}\phi_n = \lambda_n \phi_n . \quad \text{Define } \psi_n \equiv \frac{\not{D}\phi_n}{\sqrt{\lambda_n}} , \ \lambda_n > 0.$$

Then, $\not{D}\bar{\not{D}}\psi_n = \not{D}\bar{\not{D}}\not{D}\dfrac{\phi_n}{\sqrt{\lambda_n}} = \lambda_n \psi_n$. $\langle \psi_n | \psi_m \rangle = \delta_{nm}$.

Second of all,

Statement We can represent Ind $\not{D}$ as

$$\text{Ind}\,\not{D} = \text{Tr}\left\{ \frac{M^2}{-\not{D}\bar{\not{D}} + M^2} - \frac{M^2}{-\bar{\not{D}}\not{D} + M^2} \right\},$$

$\downarrow$

is independent of M.

proof: $$\text{Ind}\,\not{D} = \text{Tr}\left\{ \frac{M^2}{-\not{D}\bar{\not{D}} + M^2} - \frac{M^2}{-\bar{\not{D}}\not{D} + M^2} \right\} =$$

$$= \sum_{\lambda_n \neq 0} \langle \psi_n | \frac{M^2}{-\not{D}\bar{\not{D}} + M^2} | \psi_n \rangle - \langle \phi_n | \frac{M^2}{-\bar{\not{D}}\not{D} + M^2} | \phi_n \rangle +$$

$$+ \sum_i \langle \psi_i^0 | \psi_i^0 \rangle - \sum_j \langle \phi_j^0 | \phi_j^0 \rangle =$$

$$= \sum_{\lambda_n \neq 0} \frac{M^2}{-\lambda_n + M^2} \left\{ \langle \psi_n | \psi_n \rangle - \langle \phi_n | \phi_n \rangle \right\} + \sum_i 1 - \sum_j 1 =$$

$$= \dim\ker \not{D} - \dim\ker \bar{\not{D}} \quad \bullet$$

Finally, we represent it as

$$\boxed{\text{Ind}\,\not{D} = \text{Tr}\left\{ \frac{M^2}{-\not{\mathcal{D}}^2 + M^2} \gamma_5 \right\}}$$

$$\not{\mathcal{D}} = \begin{pmatrix} 0 & -i\not{D} \\ i\bar{\not{D}} & 0 \end{pmatrix} \qquad \gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

(1.4) Calculating the index of the Dirac operator

Statement

$$\boxed{\mathrm{Ind}\,\not D = \frac{1}{16\pi^2}\int d^4x\, \mathrm{tr}\, F_{\mu\nu} * F_{\mu\nu}}$$

$$F_{\mu\nu} = F^a_{\mu\nu} T_a \quad ; \quad \mathrm{tr}\,(T_a T_b) = -\delta_{ab}\, T(R)$$

proof.

$$\mathrm{Ind}\,\not D = \int d^4x \int \frac{d^4k\, d^4k'}{(2\pi)^8}\, \mathrm{tr}\, \langle x|k'\rangle\langle k'|\frac{M^2}{-\not D^2+M^2}\gamma_5|k\rangle\langle k|x\rangle \;(=)$$

tr denotes the trace over group indices and spinor indices ; $\langle k|x\rangle \sim e^{-ikx}$

$$(=) \int d^4x \int \frac{d^4k}{(2\pi)^4}\, \mathrm{tr}\left\{\frac{M^2}{-(-i\not k+\not D)^2+M^2}\gamma_5\right\}.$$

Rewriting

$$-(-i\not k+\not D)^2+M^2 = (k^2+M^2) - \left(-2ik\mathcal{D} + \mathcal{D}_\mu^2 + \frac{1}{2}\gamma_\mu\gamma_\nu F_{\mu\nu}\right)$$

and rescaling $k_\mu \to M k_\mu$, we get

$$\mathrm{Ind}\,\not D = \int d^4x\, M^4 \int \frac{d^4k}{(2\pi)^4}\, \mathrm{tr}\left\{\frac{1}{(1+k^2) - \left(-\frac{2ik\mathcal{D}}{M} + \frac{\mathcal{D}^2}{M^2} + \frac{1}{2}\frac{\gamma_\mu\gamma_\nu F_{\mu\nu}}{M^2}\right)}\gamma_5\right\} =$$

$$= |M\to\infty| = \int d^4x \int \frac{d^4k}{(2\pi)^4}\frac{1}{(1+k^2)^3}\frac{1}{2^3}\, \mathrm{tr}\{\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5 F_{\mu\nu}F_{\rho\sigma}\} =$$

$$= \int d^4x\, \frac{1}{32\pi^2}\, \mathrm{tr}\,(T_a T_b)\, \epsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu} F^b_{\rho\sigma} =$$

$$= \frac{1}{16\pi^2}\int d^4x\, \mathrm{tr}\,(F_{\mu\nu} * F_{\mu\nu}) .$$

- We defined K for the fundamental repr.

So, $\mathrm{tr}(T_a T_b) = -\frac{1}{2}\delta_{ab}$.

to find the number of zero modes for anti-instantons, we worked with fermions in the adjoint representation for which we have that $\mathrm{tr}(T_a T_b) = -\delta_{ab} N$

So, $\mathrm{Ind}\, \not{D} = 2\left(\frac{1}{32\pi^2}\int d^4x\, F^a_{\mu\nu} * F^b_{\mu\nu}\right)\mathrm{tr}\,T_a T_b = 2|k|N$

Finally.. Anti instantons have twice as many bosonic collective coordinates as there are fermionic zero modes in the adjoint representation.

Dimension of moduli space of anti instanton with charge K and group $SU(N)$ is $4N|k|$

② Self dual equations in terms of complex coordinates

x_1, x_2, x_3, x_4 parametize $\mathbb{R}^4$

$$\begin{cases} z_1 = x_2 + i x_1 \\ z_2 = x_4 + i x_3 \end{cases} \qquad \begin{cases} \mathcal{D}_1 = \frac{1}{2}(D_2 - i D_1) \\ \mathcal{D}_2 = \frac{1}{2}(D_4 - i D_3) \end{cases}$$

Then, anti self duality condition

$$F_{\mu\nu} = -*F_{\mu\nu} \Longrightarrow \boxed{\begin{cases} [\mathcal{D}_1, \mathcal{D}_2] = 0 = [\mathcal{D}_1^\dagger, \mathcal{D}_2^\dagger] \\ [\mathcal{D}_1, \mathcal{D}_1^\dagger] + [\mathcal{D}_2, \mathcal{D}_2^\dagger] = 0 \end{cases}}$$

③ The map $\gamma : \mathbb{CP}^3 \to S^4$ and self duality equations

In this subsection, we will show that if we want to find solutions of the anti-self-duality equations in $\mathbb{R}^4$, we can consider the connections in $\mathbb{C}^4$ lifted from $\mathbb{R}^4$ with the map γ for which the curvature G_{ij} in $\mathbb{C}^4$ becomes zero: $G_{ij}=0$.

There is a natural map $\gamma: \mathbb{C}^4 \to S^4$ if we use the fact that $\boxed{\mathbb{H}P^1 \cong S^4}$ where $\mathbb{H}P'$ is quaternion projective line defined as

$$\mathbb{H}P' = \{(q_1, q_2) \in \mathbb{H}^2 \mid (q_1, q_2) \sim (qq_1, qq_2) \text{ with } q \in \mathbb{H}^*\}.$$

where above we represent quaternion nubmers $\mathbb{H}$ as $\boxed{\mathbb{H} = \mathbb{C} + \mathbb{C}j}$, i.e $q \in \mathbb{H} \Leftrightarrow q = z + wj$, $z, w \in \mathbb{C}$.

Why S^4? 1) $\mathbb{H}P'$ contains elements (q_1, q_2) with $q_2 \neq 0$. if we take $q_2 = 1$, we can identify this „part" of $\mathbb{H}P'$ with $\mathbb{H} \sim \mathbb{R}^4$

2) if $q_2 = 0$, we have $(q_1, 0)$ or $(1, 0)$ - the point „at infinity".

So, $\mathbb{H}P' \cong \mathbb{H} + \{\infty\} \cong \mathbb{R}^4 + \{\infty\} \cong S^4$.

The map γ. The homogeneous coordinates on $\mathbb{H}P'$ has the form $(q_1 : q_2)$ $\begin{cases} q_1 = z_1 + z_2 j \\ q_2 = z_3 + z_4 j \end{cases}$

We can replace the pair $(q_1 : q_2)$ with 4-tuple (z_1, z_2, z_3, z_4) in which not all z_i are zero.

So, we have map $\gamma: (z_1, z_2, z_3, z_4) \longrightarrow (z_1 + z_2 j : z_3 + z_4 j)$

$$\mathbb{C}^4 \longrightarrow \mathbb{H}P' \sim S^4$$

Moreover, this map reduces to $\mathbb{C}P^3 = \mathbb{C}^4 / \mathbb{C}^\times$, since

$$\lambda(z_1, z_2, z_3, z_4) \longrightarrow (\overbrace{z_1 + z_2 j}^{q_1} : \overbrace{z_3 + z_4 j}^{q_2}) \quad \text{with } \lambda \in \mathbb{C}^\times$$

So, homogeneous coordinates on $\mathbb{C}P^3$ are $(z_1 : z_2 : z_3 : z_4)$.

We obtained a map

$$\gamma : \mathbb{C}P^3 \longrightarrow \mathbb{H}P^1 \cong S^4$$

$$(z_1 : z_2 : z_3 : z_4) \longrightarrow (q_1 : q_2)$$

What about the fibre F of such a map γ ?

$$F = \gamma^{-1}(q_1 : q_2) = \{ (z'_1 : z'_2 : z'_3 : z'_4) \mid \gamma(z'_1, z'_2, z'_3, z'_4) = (q_1 : q_2) \}$$

the fibre contains the pair of the form (qz_1, qz_2, qz_3, qz_4) and $(q\lambda z_1, q\lambda z_2, q\lambda z_3, q\lambda z_4)$ where $q \in \mathbb{H}^\times$ and $\lambda \in \mathbb{C}^\times$

which under γ gives the same point $(q_1 : q_2)$.

It means that we can identify $q \sim q\lambda$. In other words if

$q = w_1 + j w_2$ with $w_1, w_2 \in \mathbb{C}$

$$(w_1 + j w_2) \sim (w_1 + j w_2)\lambda \Rightarrow (w_1, w_2) \sim \lambda(w_1, w_2)$$

which is $\mathbb{C}P^1$.

The fibre of γ is $\mathbb{C}P^1 \cong S^2$

Finally, we get that $\mathbb{C}P^3$ is $\mathbb{C}P^1$ bundle over S^4 with the projection map

$$\gamma : (z_1, z_2, z_3, z_4) \longrightarrow \hat{x} = q_2^{-1} q_1 \in S^4$$

where we used quaternion notation

$$\hat{x} = x_\mu \sigma_\mu \,, \quad \sigma_\mu = (1, \sigma_i)$$

$$\boxed{\begin{array}{l} 1 \to \sigma_0 \\ i \to \sigma_1 \\ j \to \sigma_2 \\ k \to \sigma_3 \end{array}}$$

$$q_1 = \begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{bmatrix} \qquad q_2 = \begin{bmatrix} z_3 & z_4 \\ -\bar{z}_4 & \bar{z}_3 \end{bmatrix} .$$

Also, we need to understand how j acts on $\mathbb{C}P^3$, using the property $z j = j \bar{z}$

On $\mathbb{H}P^1$ $\big(j(z_1 + z_2 j),\, j(z_3 + z_4 j) \big) =$

$$= \left(-\bar{z}_2 + \bar{z}_1 j \,,\, -\bar{z}_4 + \bar{z}_3 j \right)$$

So, we have that j induces a map σ on $\mathbb{C}P^3$

$$\boxed{\sigma(z_1, z_2, z_3, z_4) = (-\bar{z}_2, \bar{z}_1, -\bar{z}_4, \bar{z}_3)}$$

An explicit formula for x_μ has the form:

$$\begin{cases} x_1 = \dfrac{1}{2i} \dfrac{z_1\bar{z}_4 - \bar{z}_1 z_4 + z_2\bar{z}_3 - \bar{z}_2 z_3}{|z_3|^2 - |z_4|^2} \\[2ex] x_2 = \dfrac{1}{2} \dfrac{z_2\bar{z}_3 + \bar{z}_2 z_3 - z_1\bar{z}_4 - \bar{z}_1 z_4}{|z_3|^2 + |z_4|^2} \\[2ex] x_3 = \dfrac{1}{2i} \dfrac{z_1\bar{z}_3 - \bar{z}_1 z_3 + z_2\bar{z}_4 - \bar{z}_2 z_4}{|z_3|^2 + |z_4|^2} \\[2ex] x_4 = \dfrac{1}{2} \dfrac{z_1\bar{z}_3 + \bar{z}_1 z_3 + z_2\bar{z}_4 + \bar{z}_2 z_4}{|z_3|^2 + |z_4|^2} \end{cases}$$

Under the map γ, the connection on $\mathbb{R}^4$ could be lifted to $\mathbb{C}^4$

$$C_i(z) = \frac{\partial x^\mu}{\partial z_i} A_\mu(x(z))$$

The curvature has the form

$$G_{ij}(z) = \frac{\partial x^\mu}{\partial z_i}\frac{\partial x^\nu}{\partial z_j} F_{\mu\nu} = \partial_i C_j - \partial_j C_i + [C_i, C_j] =$$

$$= \frac{D(x^\mu, x^\nu)}{D(z_i, z_j)} F_{\mu\nu}$$

where
$$\frac{D(x^\mu, x^\nu)}{D(z_i, z_j)} = \frac{1}{2}\left[\frac{\partial x^\mu}{\partial z_i}\frac{\partial x^\nu}{\partial z_j} - \frac{\partial x^\nu}{\partial z_i}\frac{\partial x^\mu}{\partial z_j}\right]$$

and it is „self dual"

$$\frac{D(x^\mu, x^\nu)}{D(z_i, z_j)} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\frac{D(x^\rho, x^\sigma)}{D(z_i, z_j)}$$

From this property it follows that

$$\boxed{G_{ij} = 0}$$

Since $G_{ij} = \underbrace{\frac{D(x^\mu, x^\nu)}{D(z_i, z_j)}}_{\text{„self dual"}} \underbrace{F_{\mu\nu}}_{\text{anti self dual.}}$

Similarly, $\boxed{G_{\bar{i}\bar{j}} = 0}$

Summary

① Derived that $\dim I_{N,k} = 4Nk$

② $[D_1, D_2] = 0$

$[D_1, D_2^+] + [D_2, D_2^+] = 0$

③ $\gamma : \mathbb{CP}^3 \longrightarrow S^4$

$$A_\mu : F_{\mu\nu} = -*F_{\mu\nu} \Longrightarrow C_i(z) = 0 \begin{cases} G_{ij} = 0 \\ G_{\bar{i}\bar{j}} = 0 \end{cases}$$

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