The quantum method of separation of variables by the example of open two-particle Toda chain
Квантовый метод разделения переменных на примере открытой двухчастичной цепочки Тоды

Pavel Antonenko

May 2, 2020

1 Introduction

The quantum N-particle Toda chain is a one-dimensional quantum-mechanical system of equal particles in which the interaction is described by the Hamiltonian [1, 2]

$$\hat{H}_N(\alpha) = \sum_{n=1}^{N} \frac{\hat{p}_n^2}{2} + \sum_{n=1}^{N-1} e^{\hat{x}_{n+1}-\hat{x}_n} + \alpha e^{\hat{x}_1-\hat{x}_N}.$$

$\hat{p}_n$, $\hat{x}_n$ are the operators of momentum and coordinate of n^{th} particle. Bond length, bond energy, Planck constant and mass of a particle are chosen to be the units. $\alpha \geq 0$ is a parameter which determines the interaction between the "periphery" particles. $\alpha = 1$ gives the periodic and $\alpha = 0$ the open Toda chain.

Quantum Toda chain is a quantum integrable system, i.e. one can explicitly construct for it a complete set of observables using the formalism of the Quantum Spectral Transform Method [2]. The common eigenfunctions of this set of observables for N particles can be represented in terms of eigenfunctions for $N-1$ particles. This reduction is called the quantum method of separation of variables [2]. In order to demonstrate this method by the example of Toda chain, the simplest case of open chain consisting of two particles is considered.

In section 2 the eigenstates of Hamiltonian are found by the direct solution of Schrödinger equation. The solutions are found in terms of Macdonald function. In section 3 L-operator and monodromy matrix, the notions of Quantum Spectral Transform Method, are introduced. They are used to construct the complete set of observables, which is contained in one element of the monodromy matrix. In section 4 the eigenproblem for this matrix element is solved, Mellin-Barnes and Gauss-Givental integral representations for the wavefunction of two-particle chain in terms of one-particle wavefunction are obtained. Both of these representations can be generalized to arbitrary number of particles [3, 4]. In section 5 the equivalence between solutions from section 2 and section 4 is proven. In section 6 the completeness and ortogonality of the obtained set of eigenstates is shown. In Conclusion the results are summed up and reality of the quantum numbers parametrising eigenfunctions in Mellin-Barnes and Gauss-Givental representations is discussed.

2 Direct solution of Schrödinger equation

Two-particle Hamiltonian of the open Toda chain in coordinate representation is given by

$$\hat{H} = -\frac{1}{2}(\partial_1^2 + \partial_2^2) + e^{x_1-x_2},$$

where $\partial_k = \frac{\partial}{\partial x_k}$. By the usage of commutation relations $[-i\partial_k, x_l] = -i\delta_{kl}$ it can be easily verified that $\hat{H}$ commutes with the total momentum operator $\hat{P} = -i(\partial_1+\partial_2)$. Since in the case under consideration the system is of two degrees of freedom, $(\hat{H},\hat{P})$ is a complete set of observables. The system of equations for their common eigenstates is

$$\begin{cases} \left[-(\partial_1^2+\partial_2^2)/2+e^{x_1-x_2}\right]\Psi(x_1,x_2)=E\,\Psi(x_1,x_2) \\ -i(\partial_1+\partial_2)\Psi(x_1,x_2)=P\,\Psi(x_1,x_2) \end{cases}.$$

The change of variables $Q=\frac{1}{2}(x_1+x_2)$, $q=\frac{1}{2}(x_1-x_2)$ leads to the equivalent system:

$$\begin{cases} \left[-(\partial_Q^2+\partial_q^2)/4+e^{2q}\right]\Psi(Q,q)=E\,\Psi(Q,q) \\ -i\partial_Q\Psi(Q,q)=P\,\Psi(Q,q) \end{cases}. \tag{1}$$

It follows from the second equation in (1) that

$$\Psi(Q,q)=e^{iPQ}\varphi(q). \tag{2}$$

Substituting (2) into the first equation in (1) one can obtain the following equation for $\varphi(q)$:

$$[-\partial_q^2+4e^{2q}]\varphi(q)=\lambda^2\varphi(q), \tag{3}$$

where $\lambda^2=4E-P^2$. (3) can be interpreted as one-dimensional Schrödinger equation in exponential potential $4e^{2q}$ (figure 1).

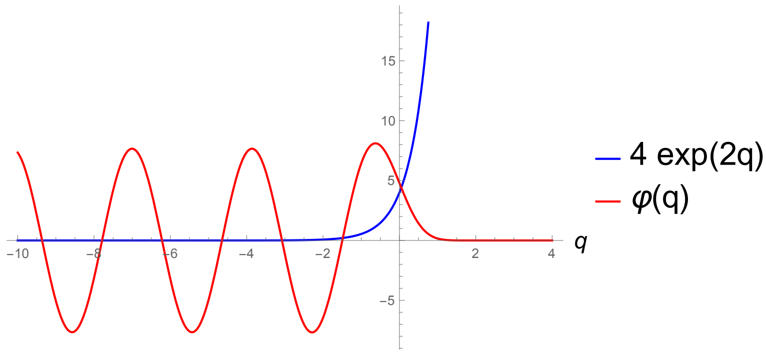

Рис. 1: exponential potential and the plot of $\varphi(q)$.

From the plot of the potential energy it follows that $\varphi(q)$ becomes the free particle wavefunction when $q\to-\infty$, and must tend to zero behind the exponential barrier, i.e. when $q\to+\infty$. By the change of variable $z=2e^q$ (3) can be transformed into

$$\left[(z\partial_z)^2-(z^2+(i\lambda)^2)\right]\varphi(z)=0. \tag{4}$$

(4) is the modified Bessel equation [5], the basis of its solutions consists of $I_{i\lambda}(z)$, $I_{-i\lambda}(z)$, where the Infeld function I_ν is given by

$$I_\nu(z)=\sum_{k=0}^{\infty}\frac{(z/2)^{2k+\nu}}{k!\Gamma(k+\nu+1)},\quad \nu\in\mathbb{C}. \tag{5}$$

However, the only linear combination of $I_{i\lambda}$ and $I_{-i\lambda}$ which fullfils the physical condition of decaying at $z\to+\infty$ (i.e. $q\to+\infty$) is the Macdonald function [5]

$$K_{i\lambda}(z)=\frac{\pi}{2\sin(\pi i\lambda)}(I_{-i\lambda}(z)-I_{i\lambda}(z)),\qquad K_{i\lambda}(z)\sim\sqrt{\frac{\pi}{2z}}\,e^{-z},\ z\to+\infty.$$

As a result, the eigenfunctions of the open two-particle Toda chain Hamiltonian are

$$\Psi(x_1,x_2|P,\lambda)=\sqrt{\frac{\lambda\sinh(\pi\lambda)}{4\pi^3}}\,e^{iPQ}K_{i\lambda}(2e^q),\quad Q=\frac{1}{2}(x_1+x_2),\quad q=\frac{1}{2}(x_1-x_2). \tag{6}$$

The normalization factor $\sqrt{\frac{\lambda\sinh(\pi\lambda)}{4\pi^3}}$ has not been derived, it will be derived in section 6.

3 L-operator, monodromy matrix, and the complete set of observables

To the k^{th} particle in the chain one can match the operator

$$L_k(u) = \begin{pmatrix} u - \hat{p}_k & e^{-\hat{x}_k} \\ -e^{\hat{x}_k} & 0 \end{pmatrix}. \tag{7}$$

It is called the L-operator [2]. $L_k(u)$ contains the operators of physical quantities corresponding to k^{th} particle and depends on the complex parameter u called the spectral parameter. It acts on the tensor product of quantum space of Toda chain and two-dimensional auxiliary space. On the auxiliary space it acts as 2×2 matrix in (7).

The monodromy matrix $T(u)$ of the N-particle system is the product of L-operators of all particles with the same auxiliary space [2]:

$$T_N(u) = L_N(u)L_{N-1}(u)\ldots L_1(u) = \begin{pmatrix} A_N(u) & B_N(u) \\ C_N(u) & D_N(u) \end{pmatrix}.$$

It can be proven by induction that the operator $A_N(u)$ is a polynomial of degree N in u and posseses the complete set of observables as coefficients of different powers of u [2], and the Hamiltonian can be represented in terms of these operators.

In the case of two particles

$$A_2(u) = u^2 - \hat{P}u + \hat{P}^2/2 - \hat{H}.$$

As mentioned above, $(\hat{P}, \hat{P}^2/2 - \hat{H})$ form a complete set of observables as coefficients of powers of u. Expressing $\hat{H}$ in terms of these two operators, it is easy to deduce that $(\hat{H}, \hat{P})$ is also a complete set of observables, which is in agreement with section 2.

Since the coefficients of all powers of u in $A_N(u)$ are simultaneously diagonalizable, the problem of determination of their common eigenvectors, which are also the eigenvectors of the Hamiltonian, is equivalent to the eigenproblem for $A_N(u)$. For $N = 2$ this problem reads

$$A(u)\Psi = [u^2 - Pu + P^2/2 - E]\Psi = (u - \lambda_1)(u - \lambda_2)\Psi,$$

where P is the total momentum, E is the energy, the quantum numbers λ_1 and λ_2 are the roots of the polynomial $u^2 - Pu + P^2/2 - E$ in u. P and E can be expressed in terms of λ_1, λ_2:

$$P = \lambda_1 + \lambda_2, \quad P^2/2 - E = \lambda_1\lambda_2, \tag{8}$$

and vice versa.

As a result, in the case of two particles we have the following equation for the eigenfunctions of the Hamiltonian:

$$\underbrace{[(u + i\partial_2)(u + i\partial_1) - e^{x_1 - x_2}]}_{A_2(u)}\Psi(x_1, x_2|\lambda_1, \lambda_2) = (u - \lambda_1)(u - \lambda_2)\Psi(x_1, x_2|\lambda_1, \lambda_2). \tag{9}$$

In the next section this equation will be solved using two spesial integral representations of the wavefunction Ψ – Mellin-Barnes and Gauss-Givental representations, which will lead to the expression of a wavefunction of two-particle system in terms of the integral of one-particle wavefunction. This reduction to the system with one particle less can be generalized to the case of Toda chain with arbitrary number of particles [3, 4].

4 Integral representations of wavefunctions

One can try to find the solution of (9) using an analogue of the Fourier transform:

$$\Psi(x_1, x_2|\lambda_1, \lambda_2) = \int\limits_{C_1}\int\limits_{C_2} d\gamma_1\, d\gamma_2\, e^{i\gamma_1 x_1} e^{i\gamma_2 x_2} F(\gamma_1, \gamma_2|\lambda_1, \lambda_2), \tag{10}$$

where the contours C_1 and C_2 go in parallel with the real axis. Substituting the expression for Ψ from (10) into (9) one obtains the following expression:

$$\int\limits_{C_1}\int\limits_{C_2} d\gamma_1\, d\gamma_2\, (u-\gamma_1)(u-\gamma_2)e^{i\gamma_1 x_1}e^{i\gamma_2 x_2}F(\gamma_1,\gamma_2|\lambda_1,\lambda_2)-$$

$$-\int\limits_{C_1}\int\limits_{C_2} d\gamma_1\, d\gamma_2\, e^{i(\gamma_1-i)x_1}e^{i(\gamma_2+i)x_2}F(\gamma_1,\gamma_2|\lambda_1,\lambda_2)=$$

$$=\int\limits_{C_1}\int\limits_{C_2} d\gamma_1\, d\gamma_2\, (u-\lambda_1)(u-\lambda_2)e^{i\gamma_1 x_1}e^{i\gamma_2 x_2}F(\gamma_1,\gamma_2|\lambda_1,\lambda_2). \quad (11)$$

In order to group all the summands in (11) into one integral and obtain the equation for $F(\gamma_1,\gamma_2|\lambda_1,\lambda_2)$, the second integral in the L.H.S. of (11) must be transformed to expression with the exponentials of the same form as they appear in two other integrals. Using the change of variables $\gamma_1'=\gamma_1-i$, $\gamma_2'=\gamma_2+i$ one obtains

$$\int\limits_{C_1}\int\limits_{C_2} d\gamma_1\, d\gamma_2\, e^{i(\gamma_1-i)x_1}e^{i(\gamma_2+i)x_2}F(\gamma_1,\gamma_2|\lambda_1,\lambda_2)=$$

$$=\int\limits_{C_1-i}\int\limits_{C_2+i} d\gamma_1'\, d\gamma_2'\, e^{i\gamma_1' x_1}e^{i\gamma_2' x_2}F(\gamma_1'+i,\gamma_2'-i|\lambda_1,\lambda_2)=$$

$$=\int\limits_{C_1}\int\limits_{C_2} d\gamma_1'\, d\gamma_2'\, e^{i\gamma_1' x_1}e^{i\gamma_2' x_2}F(\gamma_1'+i,\gamma_2'-i|\lambda_1,\lambda_2). \quad (12)$$

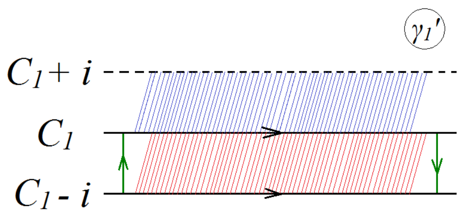

Рис. 2: contours C_1, C_1-i and C_1+i in the plane of the complex variable γ_1'.

For the correctness of the last equality in (12) $F(\gamma_1'+i,\gamma_2'-i|\lambda_1,\lambda_2)$ must obey two conditions. First, it must have no poles in variable γ_1' in the band between the contours C_1 and C_1-i (red-shaded in the figure 2). Equivalently, $F(\gamma_1,\gamma_2|\lambda_1,\lambda_2)$ must have no poles in variable γ_1 in the band between the contours C_1 and C_1+i (blue-shaded in the figure 2). This condition will define the position of C_1 in relation to the points λ_1 and λ_2. Second, integrals along the green segments between C_1 and C_1-i on the figure 2 must tend to zero when the segments are moved to infinity. The solution $F(\gamma_1,\gamma_2|\lambda_1,\lambda_2)$ will satisfy this condition.

Substituting the result of (12) into (11), gathering all the summands into one integral, and equating the integrand to zero one obtains the equation for $F(\gamma_1,\gamma_2|\lambda_1,\lambda_2)$:

$$(u-\gamma_1)(u-\gamma_2)F(\gamma_1,\gamma_2|\lambda_1,\lambda_2)-F(\gamma_1+i,\gamma_2-i|\lambda_1,\lambda_2)=(u-\lambda_1)(u-\lambda_2)F(\gamma_1,\gamma_2|\lambda_1,\lambda_2).$$

The equation for coefficients of u^2 is trivial. Equating the coefficients of u^1 one obtains

$$(\lambda_1+\lambda_2-\gamma_1-\gamma_2)F(\gamma_1,\gamma_2|\lambda_1,\lambda_2)=0,$$

therefore

$$F(\gamma_1,\gamma_2|\lambda_1,\lambda_2)=\delta(\lambda_1+\lambda_2-\gamma_1-\gamma_2)\,f(\gamma_1), \quad (13)$$

where $\delta(x)$ is the Dirac delta function, $f(\gamma_1)$ is some function of γ_1 which should be found. Equating the coefficients of u^0 one obtains

$$(\gamma_1\gamma_2-\lambda_1\lambda_2)F(\gamma_1,\gamma_2|\lambda_1,\lambda_2)=F(\gamma_1+i,\gamma_2-i|\lambda_1,\lambda_2). \quad (14)$$

Substituting expression (13) for F into (14) and using $\gamma_2 = \lambda_1 + \lambda_2 - \gamma_1$ from the dirac delta function in (13) one obtains the equivalent form of (14):

$$i(\lambda_1 - \gamma_1) i(\lambda_2 - \gamma_1) F(\gamma_1, \gamma_2 | \lambda_1, \lambda_2) = F(\gamma_1 + i, \gamma_2 - i | \lambda_1, \lambda_2). \tag{15}$$

From the property of the gamma function $\Gamma(z+1) = z\Gamma(z)$ [5] it follows that

$$F(\gamma_1, \gamma_2 | \lambda_1, \lambda_2) = \delta(\lambda_1 + \lambda_2 - \gamma_1 - \gamma_2)\, \Gamma(i(\lambda_1 - \gamma_1))\, \Gamma(i(\lambda_2 - \gamma_1)) \tag{16}$$

is a solution of (15). All that remains now is to define the position of the contour C_1 of integration with respect to γ_1. From the above formulated conditions on $F(\gamma_1, \gamma_2 | \lambda_1, \lambda_2)$, the fact that the poles of the gamma function are nonpositive integers [5] and (16) it follows that γ_1 must obey

$$\Re(i(\lambda_k - \gamma_1)) > 0 \Leftrightarrow -\Im(\lambda_k - \gamma_1) > 0 \Leftrightarrow \Im\gamma_1 > \Im\lambda_k, \quad k = 1, 2,$$

where $\Re$, $\Im$ are real and imaginary parts of a complex number. Therefore, C_1 must be located above the points λ_1 and λ_2.

As a result, the expression for the wavefunction $\Psi(x_1, x_2 | \lambda_1, \lambda_2)$ reads

$$\Psi(x_1, x_2 | \lambda_1, \lambda_2) = \int\limits_C d\gamma\, \Gamma(i(\lambda_1 - \gamma))\, \Gamma(i(\lambda_2 - \gamma))\, e^{i(\lambda_1 + \lambda_2 - \gamma)x_2} e^{i\gamma x_1}, \tag{17}$$

where C goes above the real axis. It is called the Mellin-Barnes representation [3]. There is no problem with convergence of the integral because when $|\gamma|$ tends to infinity $|\Gamma(i(\lambda_k - \gamma))|$ decays faster than any negative power of γ. As it was mentioned in section 1 and at the end of section 3, $\Psi(x_1, x_2 | \lambda_1, \lambda_2)$ is expressed in terms of an integral of one-particle wavefunction $e^{i\gamma x_1}$. It is significant that in the case of the Mellin-Barnes representation the variable of integration is the quantum number γ of $(2-1)$-particle wavefunction. This is in contrast with the Gauss-Givental representation where the variables of integration are space coordinates. The Mellin-Barnes representation can be generalized to the case of N particles, and the corresponding eigenfunctions can be expressed in terms of integrals of $(N-1)$-particle wavefunctions [3]. The integration variables are the quantum numbers of these wavefunctions.

The gamma function has the integral representation [5]

$$\Gamma(z) = \int\limits_0^\infty t^{z-1} e^{-t}\, dt.$$

The change of variable $t = e^y$ leads to the expression

$$\Gamma(i\beta) = \int\limits_{-\infty}^{+\infty} dy\, e^{i\beta y - e^y}.$$

Substituting this into (17) and making some calculations in the obtained triple integral one can deduce the Gauss-Givental representation of $\Psi(x_1, x_2 | \lambda_1, \lambda_2)$ [4]:

$$\Psi(x_1, x_2 | \lambda_1, \lambda_2) = \int\limits_{-\infty}^{+\infty} dy\, e^{i\lambda_2(x_1 + x_2 - y) - e^{x_1 - y} - e^{y - x_2}}\, e^{i\lambda_1 y}. \tag{18}$$

As in (17) it is an integral with $(2-1)$-particle wavefunction $e^{i\lambda_1 y}$. As it was already mentioned, in contrast with the Mellin-Barnes representation the integration variable is the space coordinate y. Analogously to (17) the Gauss-Givental representation can be generalized to the case of N particles, and the N-particle wavefunction can be expressed in terms of an integral of $(N-1)$-particle one where the integration variables are $N-1$ space coordinates [4].

5 The equivalence of integral representations and the expression in terms of the Macdonald function

Remembering the change of variables $Q = \frac{1}{2}(x_1 + x_2)$, $q = \frac{1}{2}(x_1 - x_2)$ from the beginning of section 2 and using (18) one obtains

$$\Psi(x_1, x_2|\lambda_1, \lambda_2) = \Psi(Q, q|\lambda_1, \lambda_2) = \int\limits_{-\infty}^{+\infty} dy\, e^{i\lambda_2(2Q-y)-e^{Q+q-y}-e^{y-Q+q}}\, e^{i\lambda_1 y} = [y' = y - Q] =$$

$$= e^{i(\lambda_1+\lambda_2)Q} \int\limits_{-\infty}^{+\infty} dy'\, e^{i(\lambda_1-\lambda_2)y'}\, e^{-2e^q \cosh y'} = e^{i(\lambda_1+\lambda_2)Q} \int\limits_{-\infty}^{+\infty} dy'\, \cosh[i(\lambda_1 - \lambda_2)y']e^{-2e^q \cosh y'}. \quad (19)$$

From (3) and (8) it follows that $\lambda_1 + \lambda_2 = P$, $\lambda_1 - \lambda_2 = \lambda$. Therefore, from (19) using the text-book integral representation for the Macdonald function [5]

$$K_\nu(z) = \int\limits_0^\infty e^{-z\cosh t} \cosh(\nu t)\, dt \quad (20)$$

one can deduce that

$$\Psi(Q, q|\lambda_1, \lambda_2) = 2e^{iPQ} K_{i\lambda}(2e^q). \quad (21)$$

Since the solutions of (9) obtained in section 4 are determined up to a coordinate-independent factor, formulas (6), (17) and (18) define one and the same eigenfunction of the Toda chain Hamiltonian up to normalization. In the next section completeness and ortogonality of the set of these eigenfunctions is shown.

6 The completeness and ortogonality of the obtained set of eigenstates

The Kontorovich-Lebedev transform and its inverse are given by

$$f(z) = \int\limits_0^\infty d\lambda\, K_{i\lambda}(z)\, g(\lambda) \quad (22)$$

and

$$g(\lambda) = \frac{2\lambda \sinh(\pi\lambda)}{\pi^2} \int\limits_0^\infty \frac{dz'}{z'}\, K_{i\lambda}(z')\, f(z'), \quad (23)$$

respectively [6]. Substituting (22) into (23) and using the change of variable $z' = 2e^q$ one can obtain the ortogonality relation for Macdonald functions

$$\frac{\lambda \sinh \lambda}{\pi^2} \int\limits_{-\infty}^\infty dq\, K_{i\lambda}(2e^q) K_{i\lambda'}(2e^q) = \frac{1}{2}\left[\delta(\lambda' - \lambda) + \delta(\lambda' + \lambda)\right]. \quad (24)$$

Substituting (23) into (22) one can obtain the completeness relation for Macdonald functions

$$\int\limits_{-\infty}^\infty d\lambda\, \frac{\lambda \sinh \lambda}{\pi^2}\, K_{i\lambda}(z) K_{i\lambda}(z') = z\delta(z' - z). \quad (25)$$

By the usage of (24) and (25) it is possible to deduce the ortogonality and completeness of the set of the open two-particle Toda chain eigenfunctions obtained in sections 2 and 4:

$$\Psi(x_1, x_2|\lambda_1, \lambda_2) = \frac{e^{iPQ}}{2\sqrt{\pi}} \sqrt{\frac{\lambda \sinh(\pi\lambda)}{\pi^2}}\, K_{i\lambda}(2e^q),$$

where $Q = \frac{1}{2}(x_1 + x_2)$, $q = \frac{1}{2}(x_1 - x_2)$, $P = \lambda_1 + \lambda_2$, $\lambda = \lambda_1 - \lambda_2$. The ortogonality of the set $\{\Psi(x_1, x_2|\lambda_1, \lambda_2)|\lambda_1, \lambda_2 \in \mathbb{R}\}$:

$$\int dx_1\, dx_2\, \overline{\Psi(x_1, x_2|\lambda_1, \lambda_2)}\Psi(x_1, x_2|\lambda_1', \lambda_2') =$$
$$= \int 2\, dQ\, dq\, \frac{e^{i(P'-P)Q}}{4\pi} \sqrt{\frac{\lambda \sinh(\pi\lambda)}{\pi^2}} \sqrt{\frac{\lambda' \sinh(\pi\lambda')}{\pi^2}}\, K_{i\lambda}(2e^q) K_{i\lambda'}(2e^q) =$$
$$= \delta(P' - P)\, \frac{1}{2} \left[\delta(\lambda' - \lambda) + \delta(\lambda' + \lambda)\right] =$$
$$= \delta(\lambda_1' + \lambda_2' - \lambda_1 - \lambda_2)\, \frac{1}{2}[\delta(\lambda_1' - \lambda_2' - \lambda_1 + \lambda_2) + \delta(\lambda_1' - \lambda_2' + \lambda_1 - \lambda_2)] =$$
$$= \frac{1}{2}[\delta(\lambda_1' - \lambda_1)\, \delta(\lambda_2' - \lambda_2) + \delta(\lambda_1' - \lambda_2)\, \delta(\lambda_2' - \lambda_1)].$$

The completeness of the set $\{\Psi(x_1, x_2|\lambda_1, \lambda_2)|\lambda_1, \lambda_2 \in \mathbb{R}\}$:

$$\int d\lambda_1\, d\lambda_2\, \overline{\Psi(x_1, x_2|\lambda_1, \lambda_2)}\Psi(x_1', x_2'|\lambda_1, \lambda_2) = \int \frac{1}{2}\, dP\, d\lambda\, \frac{e^{iP(Q'-Q)}}{4\pi}\, \frac{\lambda \sinh(\pi\lambda)}{\pi^2}\, K_{i\lambda}(2e^q) K_{i\lambda}(2e^{q'}) =$$
$$= \frac{1}{2}\, \delta(Q' - Q)\, \frac{1}{2}\, 2e^q \delta(2e^{q'} - 2e^q) = \delta(x_1' + x_2' - x_1 - x_2)\, \delta(x_1' - x_2' - x_1 + x_2) = \delta(x_1 - x_1')\delta(x_2 - x_2'). \tag{26}$$

7 Conclusion

To sum up, two integral representations of the eigenfunctions of two-particle Toda chain Hamiltonian have been obtained: the Mellin-Barnes representation

$$\Psi(x_1, x_2|\lambda_1, \lambda_2) = \int_C d\gamma\, \Gamma(i(\lambda_1 - \gamma))\, \Gamma(i(\lambda_2 - \gamma))\, e^{i(\lambda_1+\lambda_2-\gamma)x_2} e^{i\gamma x_1} \tag{27}$$

(C goes above λ_1 and λ_2 in parallel to the real axis) and the Gauss-Givental representation

$$\Psi(x_1, x_2|\lambda_1, \lambda_2) = \int_{-\infty}^{+\infty} dy\, e^{i\lambda_2(x_1+x_2-y) - e^{x_1-y} - e^{y-x_2}}\, e^{i\lambda_1 y}. \tag{28}$$

Both of them express the wavefunction of the two-particle chain in terms of an integral of one-particle wavefunction. For both representations such a reduction can be generalized to the case of arbitrary number of particles [3, 4]. The ortogonality and completeness of the set of obtained two-particle functions has been shown in section 6. From (26) it follows that the complete set is given by functions parametrized by real-valued quantum numbers λ_1, λ_2. Therefore, it is reasonable to assume that in the case of larger number of particles the corresponding quantum numbers also take real values.

References

[1] Gaudin M., The Bethe Wavefunction, Cambridge University Press, 2014.

[2] Sklyanin E.K., The quantum Toda chain, In: Sanchez N. (eds) Non-Linear Equations in Classical and Quantum Field Theory, Lecture Notes in Physics, vol 226, Springer, Berlin, Heidelberg, 1985.

[3] Kharchev S., Lebedev D., Eigenfunctions of $GL(N, \mathbb{R})$ Toda chain: Mellin-Barnes representation, Jetp Lett. 71, 235–238, 2000, arXiv:hep-th/0004065v2.

[4] Silant'ev A.V., Transition function for the Toda chain, TMF, 150:3 (2007), 371–390; Theoret. and Math. Phys., 150:3 (2007), 315–331.

[5] Abramowitz M., Stegun I. A. (Eds.), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing, New York: Dover, pp. 374-377, 1972.

[6] Bateman H., Erdelyi A., Tables of integral transforms. Volume II, McGraw-Hill Book Company, Inc., New York, Toronto, London, 1954.