Lecture 33, 12/29/2020

Friday, December 25, 2020 11:15 AM

Enumerative geometry & geometric representation theory

Start time Moscow ~~11:30~~ 18:30

New York 10:30

Monodromy of vertex functions

$|q|<1$

Monodromy of solutions to q-difference equations.

$$\psi(qz) = c\,\frac{1-az}{1-bz}\,\psi(z).$$

e.g. $\exp\left(\frac{\ln z \ln c}{\ln q}\right)$ or in ϑ-functions

$$\hookrightarrow \quad \psi(z) = \prod_{n\geqslant 0} \frac{1-q^n bz}{1-q^n az} \times \left(\text{solution of } \psi(qz) = c\,\psi(z).\right)$$

analytic in a neighbourhood of $z=0$ and meromorphic in the whole plane

Solution near $z=0$

$\longmapsto$ similarly solve equations of the form

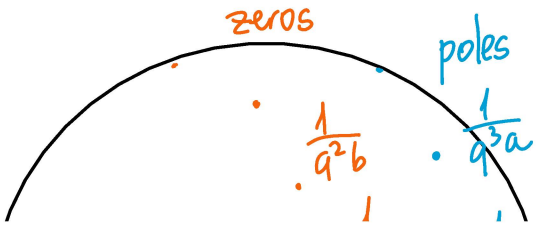
$$\psi(qz)\,M(0) = M(z)\,\psi(z).$$

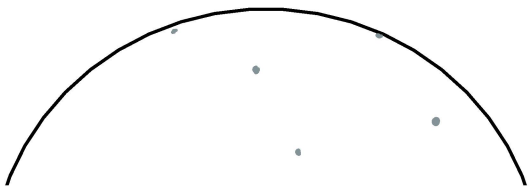
$M(0)$: constant coeff.

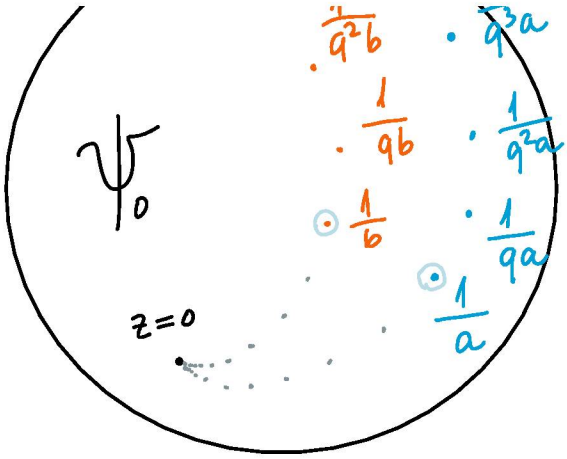
Near $z=\infty$ имеем

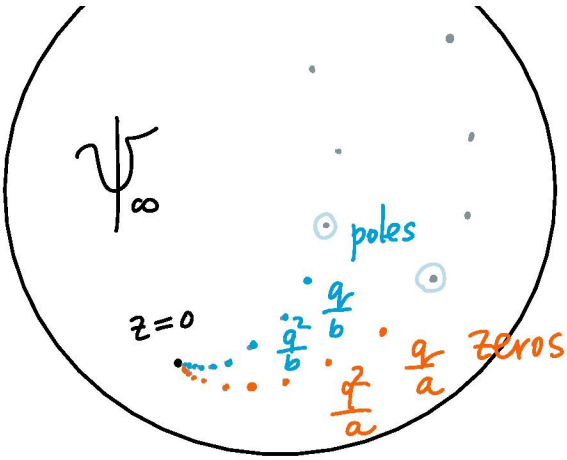
$$\psi(z) = c'\,\frac{1-q/az}{1-q/bz}\,\psi(z/q)$$

и соответственно $\psi_\infty(z) = \prod_{n\geqslant 1} \frac{1-q^n/az}{1-q^n/bz} \times$ solution of constant coeff. equation.









Monodromy: $\Psi_\infty^{-1}\Psi_0 \xrightarrow{z\mapsto qz} M(\infty)^{-1}\Psi_\infty^{-1}\Psi_0 M(0)$

section of a vector bundle over $E = \mathbb{C}^*/q$

in the abelian example:

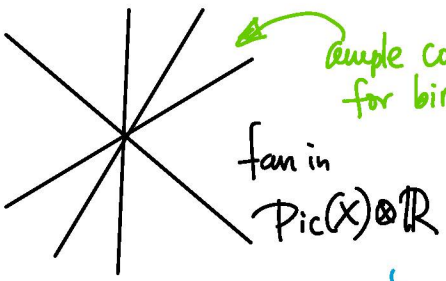
$$\Psi_\infty^{-1}\Psi_0 = \frac{\vartheta(bz)}{\vartheta(az)}$$

we will understand this in terms of elliptic cohomology

"Quantum difference equations"

for $\hbar \to 0$ defined for $z \in \mathrm{Pic}(X)\otimes\mathbb{C}^* = Z$.

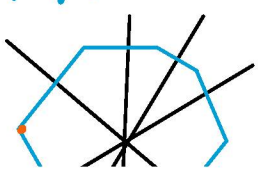
not the same Z as the ambient stack.



ample cones for birational models $\Leftarrow$ different stability conditions

fan in $\mathrm{Pic}(X)\otimes\mathbb{R}$

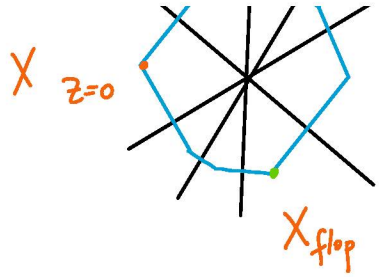
dual polytope in $\mathrm{char}(Z)\otimes\mathbb{R}$



&

toric compactification

$X_{z=0}$



toric compactification

$\overline{\mathcal{Z}} \supset \mathcal{Z}$

As a solution of q-difference equation

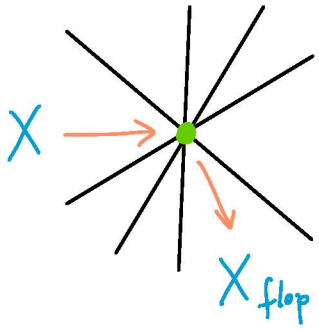
⟵o meromorphically extends to $\overline{\mathcal{Z}}$ (same for o—o = o→⟵o)

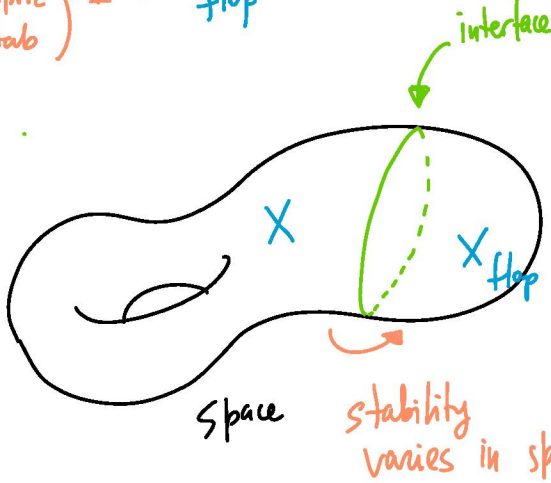
rational

Monodromy:

with Γ_q- factors

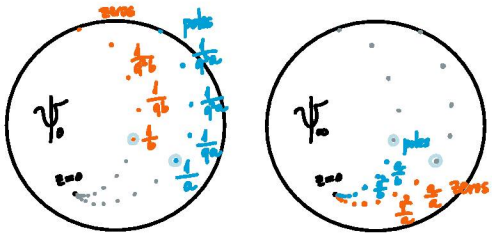
Like in Lecture #32





stability varies in space

the limit $q \to 1$, monodromy of the quantum differential equation

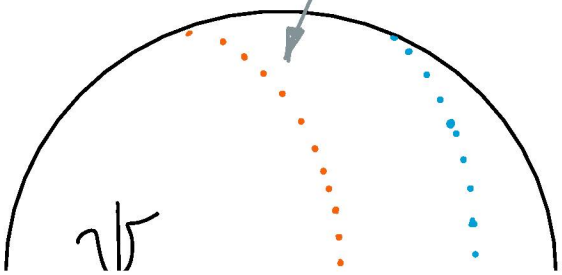


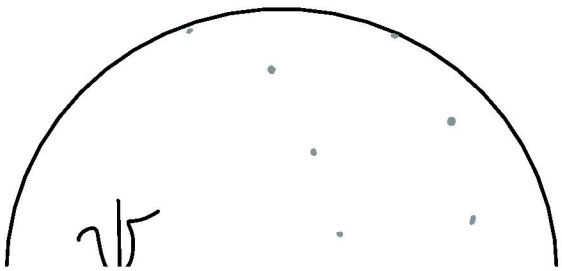
$$\prod \frac{1-q^n \hbar a}{1-q^n a} \to (1-a)^{\frac{1}{\hbar}}$$

$$\frac{1}{\hbar} = \lim \frac{\ln t}{\ln q}$$

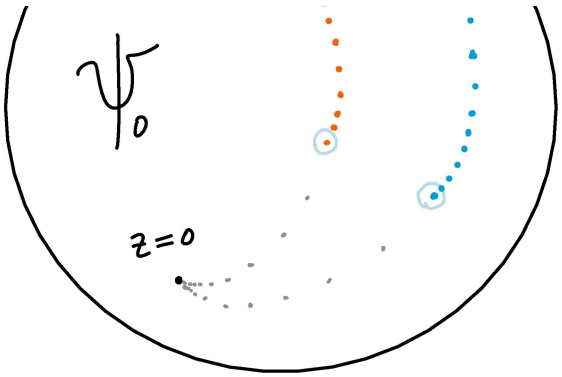
"branch cut"

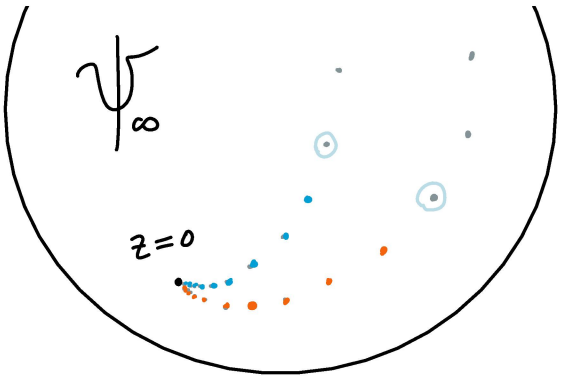
$q \to 1$.

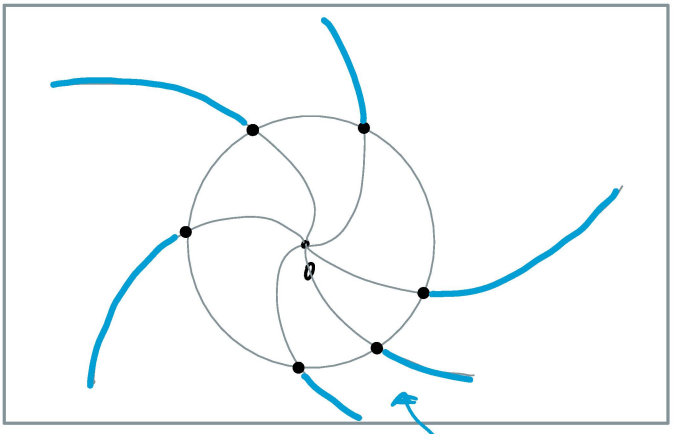




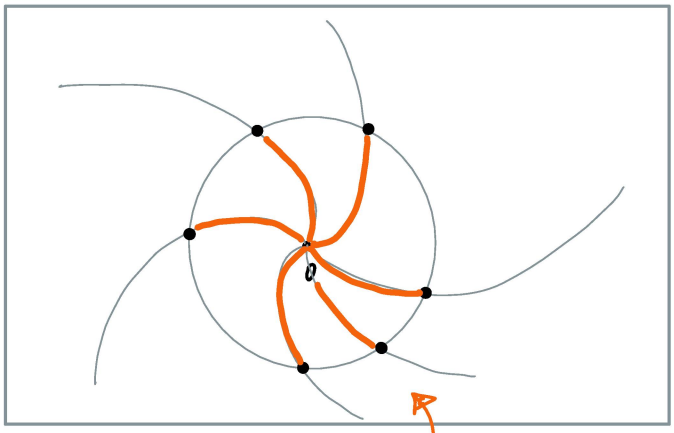
$q \to 1$.







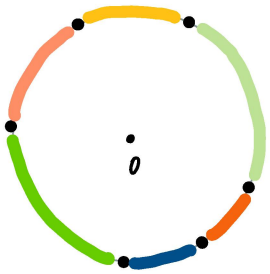
$\psi_0 \to$ solution halo



$\psi_\infty \to$ solution halo

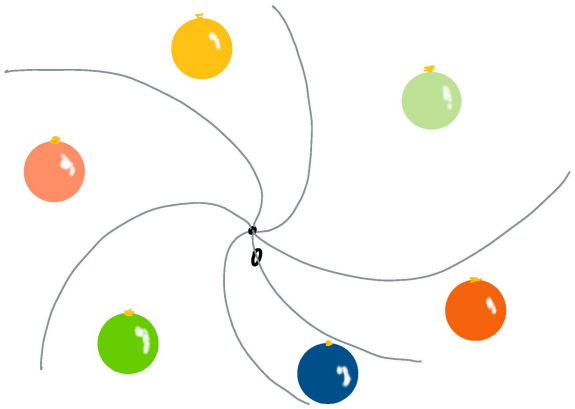
Monodromy $= \psi_\infty^{-1} \psi_0 \to$ analytic continuation

through the cut



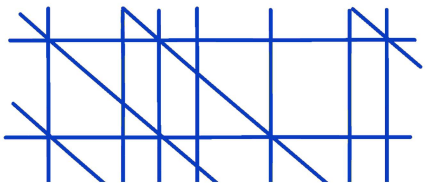
that depends on

$$\mathrm{Im}\,\frac{\ln z}{\ln q}$$



this is the compact torus in $\mathcal{Z}$ and the singularities are the Kähler roots

No walls in the elliptic word
walls appear only
in the $q \to 1$ limit



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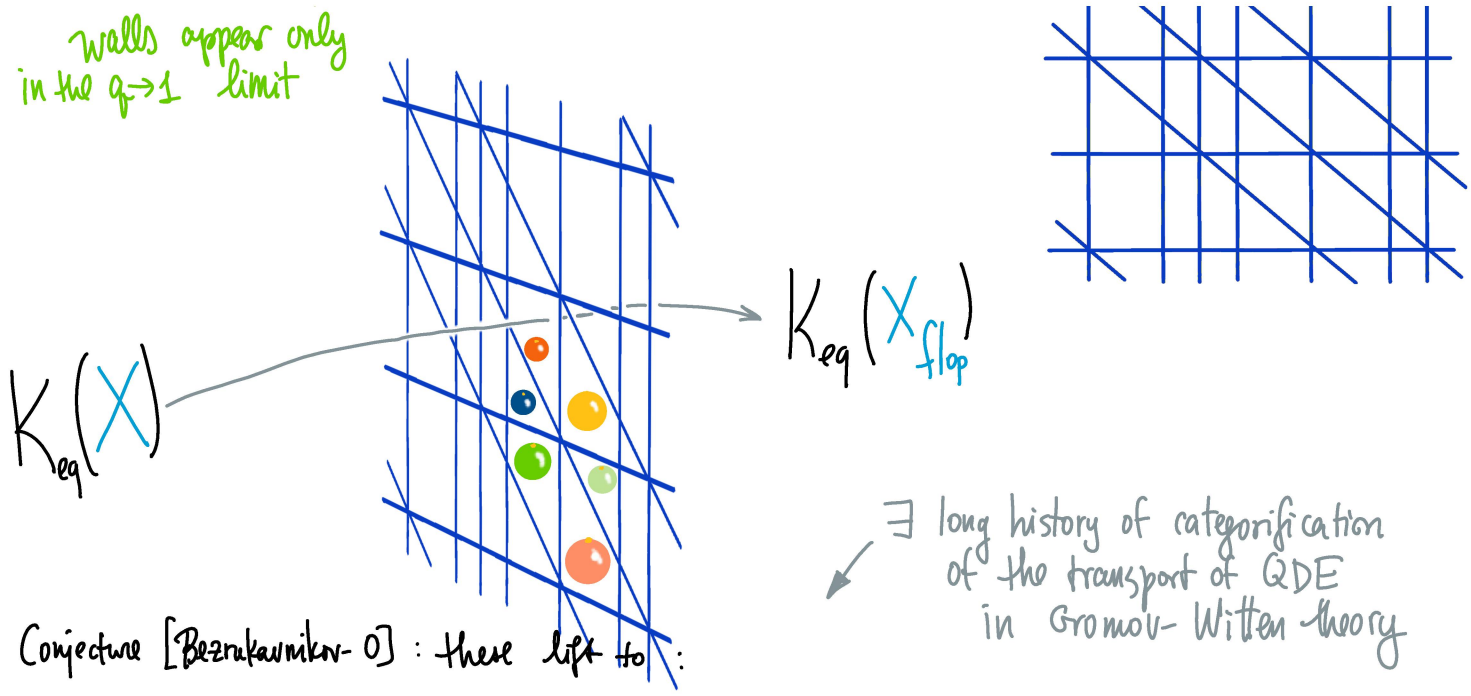


$\exists$ long history of categorification of the transport of QDE in Gromov-Witten theory

Conjecture [Bezrukavnikov-O]: these lift to:

$$D^b Coh_{eq}(X) \longrightarrow D^b Mod\ \hat{X} \longrightarrow D^b Coh_{eq}(X_{flop})$$

$\Gamma(\hat{\mathcal{O}}_X)$

quantization parameter in $\longrightarrow$ $Def\ X = H^1(T_X) \supset Pic(X)$

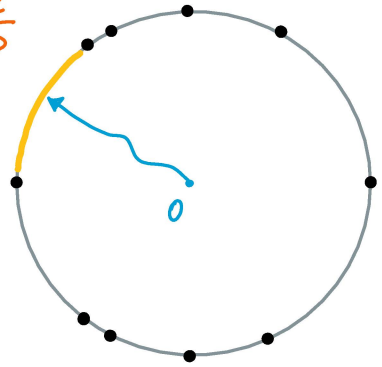
periodic modulo p

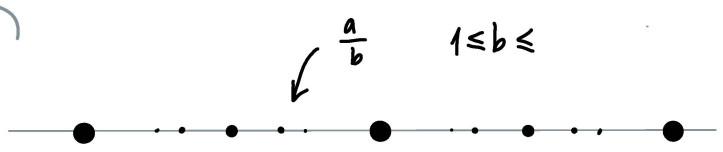
noncommutative deformation in char $p \gg 0$

e.g. for $Hilb(\mathbb{C}^2, n)$ ↗ spherical subalgebra $\subset$ rational Cherednik algebra

has a parameter c

$z = e^{2\pi i \frac{c}{p}}$





Thm: [BO] this is ✓ for $X \in$ {Nakajima varieties, T^*G/P, ...} — for these, don't have quasimaps etc.

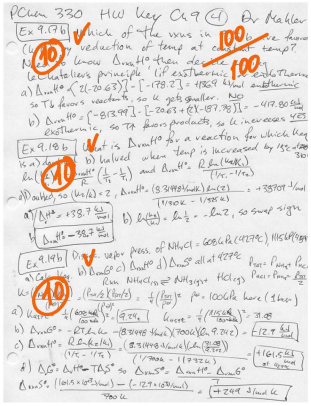
Main ingredient of proof: compatibility with $X \to X^A$, $A = torus \subset Aut(X, \omega)$

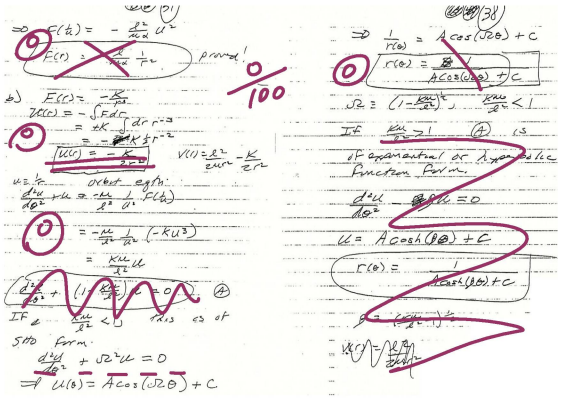
has to do with q-difference equations in equivariant variables.

what about the monodromy of those ?

A - solutions and Z - solutions

A+



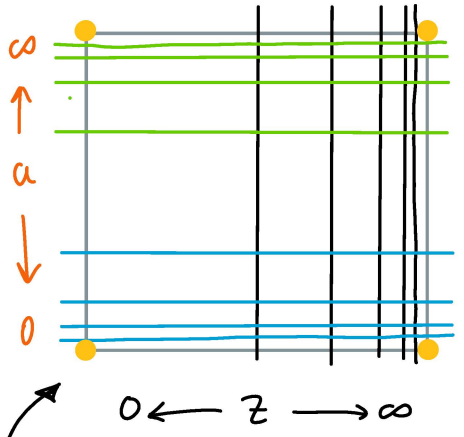


Z -

$\longleftarrow$ for $T^*\mathbb{P}^1$ $\longrightarrow$ hypergeometric function

$$\Psi_\pm = \sum_d z^d \frac{(\hbar)_d (\hbar a^{\pm 1})_d}{(q)_d (q a^{\pm 1})_d}$$

$$(x)_d = (1-x)(1-qx)\dots(1-q^{d-1}x)$$



prefactor in

||

together with the solution of the constant coeff. equation.

at all these points, the equation is <u>not</u> regular jointly in a and z

at all these points, the equation is <u>not</u> regular jointly in a and z

(1) solutions grow like $\exp\left(\frac{\ln z \ln a}{\ln q}\right)$ ← superpolynomially

(2) there is no basis of solutions holomorphic for

$$0<|z|<\varepsilon$$
$$0<|a|<\varepsilon$$

<u>z solutions</u>

$\text{prefactor}(a)\sum_d z^d\, r_d(a)$ ← $r_d(a)$ rational

Initial conditions $\Rightarrow z=0 \Rightarrow K_{eq}(X)$

<u>a - solutions</u>

$\text{prefactor}(z)\sum_k a^k\, r'_k(z)$ ← $r'_k(z)$ rational

Initial conditions $\Rightarrow a=0 \Rightarrow$ vertex functions for X^a → $K_{eq}(X^a)$

elliptic <u>pole-subtraction</u> matrix

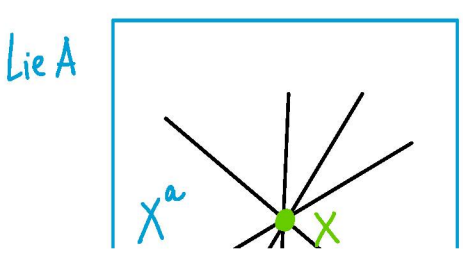
because solutions of the same q-difference equation

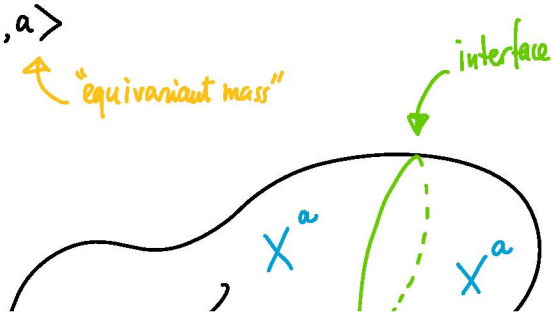
<u>Thm</u> [Aganagic-O.] this is the elliptic stable envelope

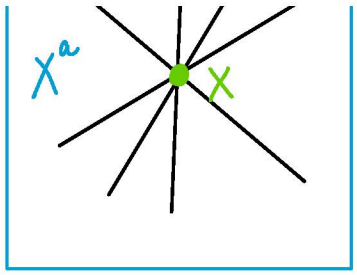
$$Ell(X^a) \to Ell(X)$$

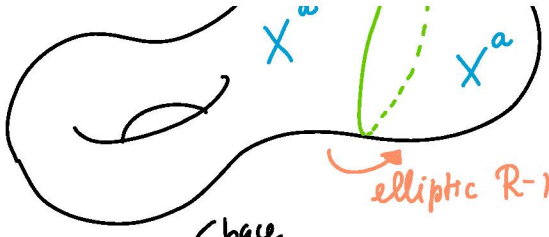
<u>Corollary</u> Monodromy in equivariant variables = Elliptic R-matrix.

Potential = ... + $\langle \mu_A, a \rangle$ ← "equivariant mass"









the equivariant masses vary in spac.

Further topics: ① 3d mirror symmetry $X \longleftrightarrow X^\vee$

exchanges interfaces of the two kinds

Useful criterion [Aganagic-O.] $X^A = \{p_i\} \longleftrightarrow \{p_i^\vee\} = (X^\vee)^{(A^\vee)}$

vertex function for the point p_i = Γ_q-function prefactors for $p_i^\vee \in X^\vee$

$a \to 0$ limit of vertex functions for X

read off $T_{p_i^\vee} X^\vee$

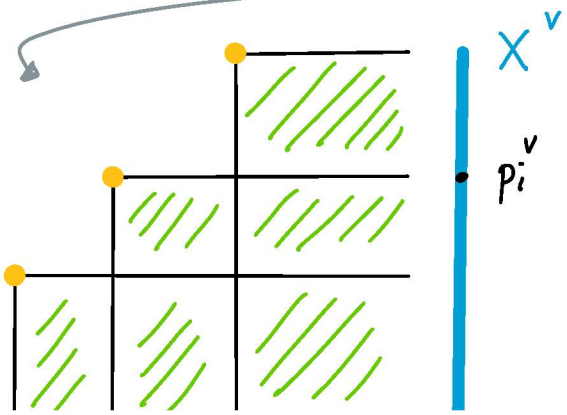
$z^\vee = a \to 0$ limit of vertex functions for $X^\vee$

$\Downarrow$

q-difference connections for X and $X^\vee$ are identified with $\begin{matrix} A & A^\vee \\ z & z^\vee \end{matrix}$

② duality interface / mother function $\mathcal{S}$

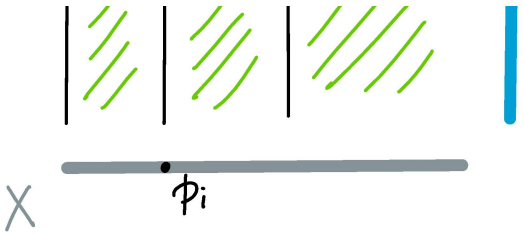
elliptic class on $X \times X^\vee$ supported on $\bigcup_i \mathrm{Attr}^f(p_i) \times \mathrm{Attr}^f(p_i^\vee)$



such that

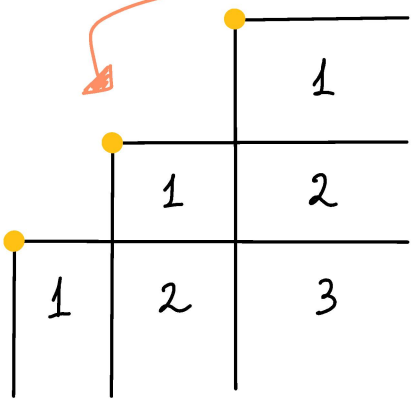
$$\mathcal{S}_{X \times \{p_i^\vee\}} = \mathrm{Stab}(p_i)$$

and vice versa

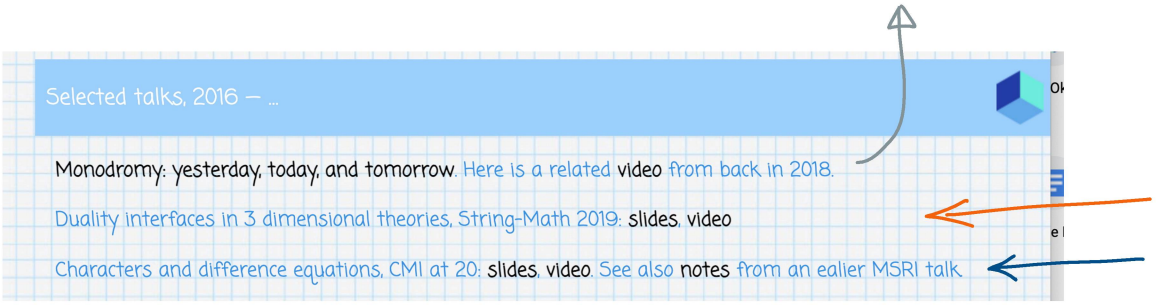


and vice versa

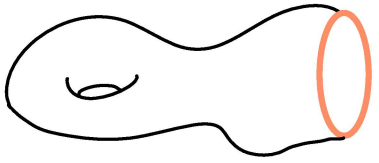
may be constructed inductively by



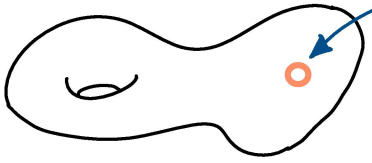
implies the equality of the monodromies and whole q-diff. connections on both sides



(3) Categorification ⇒ all these come from equivalences of category of boundary conditions



not so easy to treat in AG however,



formal loop is easy

$\mathrm{Spec}\ \mathbb{C}[t^{-1},t]] = \mathbb{D}^\times$

$QM(\mathbb{D}^\times \to X) = \widetilde{\mathrm{Loops}(X)}$

FM kernels on $\widetilde{\mathrm{Loops}(X)} \times \widetilde{\mathrm{Loops}(X^\vee)}$

universal covering

FM Kernels on Loops$(X) \times$ Loops$(X^\vee)$ universal covering

this has its own quantization $\to$ Vertex algebra in some 2D theory on the torus

↳ goes well with elliptic cohomology

Gaiotto, Braverman-Finkelberg-Nakajima

④ q-difference equations and multiplicative quantizations of X in $\sqrt{1}$.

•—○ = qq character for Verma modules for $\approx X^\vee_{mult}$

Nekrasov

← acts on $K_{crit}(QM(\mathbb{A}^1 \to X))$

especially —○

geometric understanding of how this module breaks up when $q, a \in$ roots of unity

$X = T^*\mathbb{P}^{n-1}$

$$\psi_i = \sum_{d=0}^{\infty} z^d \prod_{j=1}^{n} \prod_{k=1}^{d} \frac{1 - q^{k-1}\hbar a_j/a_i}{1 - q^k a_j/a_i}$$

singular vector = pole

⑤ KL theory for ... etc etc ...

the quantum differential equation is

- real $(z, a, \hbar) \to (\bar{z}, \bar{a}, \bar{\hbar})$.
- flop invariant
- $X_{\text{full-flop}} \simeq X$ in $\mathrm{Aut}(X, \omega)$ antiequivariant way.

real $\to 0$

$z = 1 + \varepsilon$ real

$|q| = |\hbar| = |a_i| = 1$

$q = \exp(i\varepsilon)$

$\sum_d \Rightarrow \int$

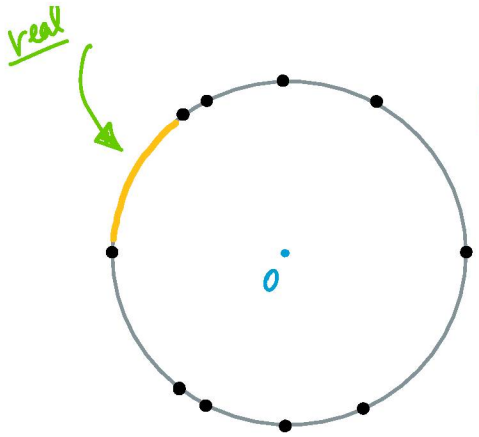
Simple modules give solutions that are real for

$(a, \hbar) \in$ compact torus $\quad z \in$ real torus

3d mirror symmetry (preserves q-difference connections).

get solution real

$\Big($ ... $\in$ unit circle.



connections).

$|z|=1$ $\quad \hbar = \exp(2\pi i\, h_{\text{cohomology}}) \in$ unit circle.

$h_{\text{cohomology}} \in \mathbb{R}$

$a = \exp(2\pi i\, a_{\text{cohomology}}) \in$ real torus.

$a_{\text{cohomology}} \in i\mathbb{R}$

$\Rightarrow$ get an involution in $\{\text{solutions of Q Differential Eq}\} \simeq K_q(X)$

antilinear w.r.t $\overline{(a, \hbar)} = (a, \hbar^{-1})$