Nonabelian stable envelopes for $\text{TI}_\text{GL}(V)$

A cohomology class for $K\text{ilb}(\alpha^4)$.

$n = 4$  $\alpha$  Fock.

\[ \bigotimes_{i=1}^n \text{Fock}(x_i) \]

Vacuum in aux.

Each is at the origin. \( \bigotimes \) both = 0.

\[ \{x_1, x_2, x_3, x_4\} \]

This matrix element of $R$-matrix is the nonabelian stable envelope.

For $X = T^* \text{Grass}(4, n)$

\[ \alpha \rightarrow K(X) \rightarrow \phi \rightarrow \bigotimes_{i=1}^n \text{C}^2(x_i) \]

\[ \phi \]

\[ T^* \text{Grass}(4, k) = \text{pt} \]

A quotient by $GL(k)$ need a sym. function in $x_1, \ldots, x_k$.

The cohomology of $\bigsqcup_{l=0}^k T^* \text{Gr}(l, k)$.
this is the point where $V \subseteq W$, i.e., $T^*\mathrm{Gr}(k, k) = \mathrm{pt}$ and all other quiver maps are $0$

\[ \phi = \{ \ldots \downarrow \downarrow \downarrow \downarrow \} \]

\[ T^*\mathrm{Gr}(0, k) \]

\[ \phi = \{ \ldots \downarrow \downarrow \downarrow \downarrow \} \]

\[ T^*\mathrm{Gr}(0, n). \]

The general case differs from by $\star = \{ W_i \hookrightarrow V_i, \text{ all other maps zero} \}$ or $W_i \hookrightarrow V_i$ depending on stability.

**General setup**

\[ X = \mathbb{Z} \sslash \mathbf{GL}(V) \]

\[ \mathbb{Z} \hookrightarrow Y = \mu^1(0) \hookrightarrow Y_{st} \longrightarrow X \]

\[ \tilde{Z} = Z \times T^* \mathrm{Hom}(V', V) \]

This is $\tilde{W}$, but it is $\dim T' = \dim V$ and $V' \cong V$ on the locus that we care about.

There is an action of $\mathbf{GL}(V')$ among loops $u \cdot L \hookrightarrow \mathbf{GL}(V)$.
Here is an action of $\text{GL}(V)$.

$$G = \text{GL}(V)$$

$$(\mathbb{C}^V)^u_{/G} \rightarrow \mathbb{Z} \times \{0\} \cup \mathbb{Z}^u \times \{V' \rightarrow V, \ V' \rightarrow \mathbb{C}^V\}$$

$$\tilde{Y} = \tilde{\mu}^{-1}(0) \quad \tilde{\mu} = \mu + AB$$

$$(A \rightarrow V') \mapsto \begin{array}{l}
B. \\
\end{array}$$

$$X = \tilde{Y}_{st} / \text{GL}(V)$$

Any point where $\mathbb{C}^V \rightarrow V$ is an isomorphism is stable
(or the other way around)

We have an action of $\text{GL}(V)$

Take stable envelope for $u' \subset$ center $\text{GL}(V)$

On the quotient by $G$

Lift to $\tilde{Y}_{st} \rightarrow$ restrict it to $\tilde{Y}_{iso} \rightarrow$ descend to $\tilde{Y}_{iso} / \text{GL}(V')$

This is the locus where $\mathbb{C}^V \rightarrow V$

$GL(V')$ acts freely

Nonabelian stable envelope was about extending a class from $Y_{st}$ to $Z$

$$Z \rightarrow \tilde{Y}_{iso} / \text{GL}(V')$$

$$\tilde{\mu} = 0$$

$$AB = -\mu$$

$$T^* \text{Hom}(\mathbb{C}^V \rightarrow V)_{iso} / \text{GL}(V')$$

The coordinate is $AB \in \text{Hom}(V, V)$.
\[ \hat{\mu} = 0 \quad \hat{\mu} = \mu + AB \]

\[ \text{Stab}_{GL(C)}(\alpha) = \text{pullback} \]
\[ \begin{array}{c}
\text{descent w.r.t. } GL(C') \\
\text{lift w.r.t. } GL(C) \end{array} \]

\( X, u \) is a component of \( \tilde{X} \)

need to check that it is supported on \( \mu^{-1}(0) \).

It is supported on \( \hat{\mu} = 0 = \mu + AB \)

follows from being supported on the attracting manifold. \[\Box\]

Reference: [Bethe eigenfunctions]

\[ K(\text{stack}) \xhookrightarrow{\delta} K(X) \]

Vertex with descendants

\[ 1 \cong \chi(\text{nonabelian elliptic stable envelope}) \quad \text{for } E = \mathbb{C}^*/q \]

\[ \mathbb{C}^* \rightarrow \mathbb{C}^*/q = E \]

induces

\[ \text{Spec } K_{eq}(X) \rightarrow \text{Spec } E_{eq}(X) \]

sheaves \leftrightarrow sheaves

more like a hypergeometric function

1) suitable \( \Gamma \)-factors

\[ \text{abbreviation from } \chi: \quad \mathbb{C} \rightarrow \mathbb{C}^* \]

really \( \chi = (\chi_{K \rightarrow E})^* \), but this is too long.
1. **Suitable \( \Gamma \)-factors**

2. **"Weak" interpretation**

\[ (\alpha, \tilde{\beta}) = \text{a function of all variables} \]

in \( K_{\text{Eq}}(\text{stack}) \)

\[ X = \mathbb{Z} \rtimes G \rightarrow \tilde{G} \rightarrow G_{\text{Ant}} \rightarrow \mathbb{1}. \]

**Theorem:**

\[ K_{G_{\text{Ant}}}(X) \xrightarrow{\text{ch(elliptic stab for } G)} K_{G}(\mathbb{Z}) \]

\[ K_{G_{\text{Ant}}}(X) \xrightarrow{\text{q-analog of Hirani map}} K_{G}(\mathbb{Z}) \]

\( K_{G}(X)_{\mathbb{Z}, \text{mero}} \) are meromorphic functions on \( \text{Spec } K_{G}(X) \times \{ |1| < 1 \} \times \text{Pic}(X) \otimes \mathbb{C}^*. \)

analytic in \( z \) in a neighborhood of \( O^* \),

\( t \) corresponds to stability condition.

\[ d = (\alpha, \text{Stab}(\beta) \frac{P}{\Gamma})_{\text{stack}} \]

\[ = \frac{1}{|W|} \int \prod_{i} \frac{dx_{i}}{\omega_{\text{min}}x_{i}} \alpha(x) \text{Stab}(\beta)(x) \frac{P}{\Gamma} \]

\( |x_{i}| = 1 \)

Last time, \( \frac{\text{Stab}(\alpha)}{Z_{x}} \rightarrow \beta = \alpha \rightarrow \beta \)

**Fundamental solution of the q-difference eq.:**
Corollary: \( (\omega, \text{Fund Solution} \cdot \beta) = \frac{1}{|W|} \int \text{d} \text{Heav}(x_i) \ f_\omega(x) \ g_\beta(x) \ \frac{\Gamma}{\Gamma} \)

- the K-theorem
- Stable envelope of \( \alpha \)
- the elliptic stable envelope of \( \beta \)

Last time \( (\omega, \text{Fund Solution} \cdot \beta) = \frac{1}{|W|} \int \text{d} \text{Heav}(x_i) \ f_\omega(x) \ g_\beta(x) \ \frac{\Gamma}{\Gamma} \)

- some contour that depends on stability condition

Immediately gives analytic continuation and monodromy of vertex functions

\( \Gamma_q \) have to with maps \( \mathbb{A}^2 \) or formal disc or \( \cdots \) to \( X \).

Maps \( (\mathbb{A}^2 \rightarrow \mathbb{A}^2) \) = polynomials \( 1, t, t^2, t^3, \ldots \) have weight \( w, w/q, w/q^2, \ldots \)

- weight of the action.

Character \( \Omega \) Maps = \( \frac{1}{(1-w^{-1})(1-qw^{-1})(1-q^2w^{-1})} \cdots \)

= \( \Gamma_q (w^{-1}) \).

- much more direct than in cohomology.

if \( S \) is a smooth stack \( \text{ev}_0 : \text{Maps} (\mathbb{A}^2 \rightarrow S) \rightarrow S \)

\( r_\alpha = \Gamma (\wedge TVC) \)
If $\mathcal{D}$ is a smooth divisor

$$e_{V_0, X} \cdot \mathcal{O}_{\text{Maps}(\mathbb{A}^2 \to S)} = \Gamma((qTV))$$

extend as multiplicative genus.

in the diagram:

$$\Gamma' = \Gamma_q(qTV \times qg + qg^*)$$

quotient

instead of $\Theta$ because it is good to work with $\mathbb{P}$ fields

$$\Gamma = \Gamma_q(\text{same but } qg^*)$$

like last time with $\frac{1}{\Delta_n}$.

Main ideas of the proof:

In the example of $\text{Hilb}(\mathbb{C}^2, 5)$

quasimaps = $\mathbb{P}^1$ moduli spaces.

\[
\begin{align*}
0_{\text{vir}}(\quad) &= \frac{ev_{\infty}(\Gamma_q \text{ expression})}{ev_0(\quad)} \\
\end{align*}
\]

not symmetrized.

from localy formula

with a tiny difference.

$$T_{\text{vir}} = H^0(...) - H^1(...) = \frac{\text{term at 0} - \text{term } \infty \cdot q^1}{1-q^1}$$

here the blue edges take
2. Let $\mathcal{F}$ be an elliptic stable envelope of some class $\mathbf{E}^\chi(x)$, section of some line bundle $(\ldots, q^3 b_1 b_2, \ldots)$.

\[ \frac{ev_0^\chi(\mathcal{F})}{ev_{\infty}^\chi(\mathcal{F})} = \text{section of some line bundle } (\ldots, b_1 b_2, \ldots) \]

\[ = \chi \quad \text{constant in } \chi \]

\[ \Rightarrow \text{turns } \mathcal{O}_{\text{vir}} \text{ into } \mathcal{O}_{\text{vir}} \otimes t^{-\chi + \text{dim} X}. \]

3. Using 1 and 2 we can move $1 \otimes \text{Stab}(\beta)$.

\[ \text{it remains to forget that the general pt here needs to be stable.} \]

\[ \bullet \bullet = \text{QM}(X) \subseteq \text{Maps}(A^1, \text{stack}). \]

\[ \text{will give the } \Gamma_q \text{ functions.} \]

\[ \text{need to extend the integration from the stable locus to the whole thing.} \]

\[ \text{the hardest part [Inductive ..., II, 3.4]} \]