Lecture 32, 12/22/2020

Monday, December 21, 2020 2:18 PM

Enumerative geometry & geometric representation theory

Start time Moscow ~~17:30~~ 18:30

New York 10:30

Nonabelian stable envelopes for $/\!/\!/\!/ \prod GL(V_i)$

a cohomology class for $\mathrm{Hilb}(\mathbb{C}^2, 4)$

vacuum in aux

α Fock. $\qquad \emptyset \qquad \bigotimes_{i=1}^{n} \mathrm{Fock}(x_i)$.

$n=4$

x_4

x_3

x_2

x_1

$\emptyset$ vacuum in "physical" space

each is at the origin. $\qquad$ both $=0$

this matrix element of R-matrix is the nonabelian stable envelope.

$\{x_1, x_2, x_3, x_4\}$.

vanishes by moment map

$V \xleftarrow{\sim} W$ $\qquad$ both $\dim V = \dim W = 4$.

vanish.

$\begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & x_3 & \\ & & & x_4 \end{pmatrix}$

via the $\simeq$ between V and W

$\rightsquigarrow$ an element of $GL(V)$

for $X = T^* \mathrm{Grass}(k, n)$

a quotient by $GL(k)$

need a symm. function in $x_1, \dots, x_k$.

$\alpha \quad K(X) \qquad \emptyset \qquad \bigotimes_{i=1}^{k} \mathbb{C}^2(x_i)$

the cohomology of $\bigsqcup_{l=0}^{k} T^*Gr(l, k)$.

$\emptyset$

$T^*Gr(k,k) = pt$

this is the point where $V \simeq W$ and all other quiver maps $=0$, i.e. $T^*Gr(k,k) = pt$

$\alpha = \{\uparrow\downarrow\downarrow\uparrow\uparrow\downarrow\downarrow\}$ k spins up out of n

$T^*Gr(k,n)$.

$\phi = \{\downarrow\downarrow\downarrow\downarrow\}$ k spins down. $T^*Gr(0,k)$

$x_1 \quad x_2 \quad x_3 \quad x_4$

$T^*Gr(k,k)$

$\{\uparrow\uparrow\uparrow\uparrow\}$.

$\phi = \{\downarrow\downarrow\downarrow\ldots\downarrow\}$ n spins down. $T^*Gr(0,n)$.

← reading backwards $B(x_1)B(x_2)B(x_3)B(x_4)|\rangle$

operator

the first and main formula of "Algebraic Bethe Ansatz" for $Y(\mathfrak{gl}_2)$, $U_q(\widehat{\mathfrak{gl}}_2)$, ...

The general case differs from by $\star = \{W_i \xrightarrow{\sim} V_i$, all other maps zero$\}$

or $W_i \xleftarrow{\sim} V_i$ depending on stability.

<u>General setup.</u>

$$X = Z /\!\!/ GL(V)$$

or $\prod GL(V_i)$

$$Z \hookleftarrow Y = \mu^{-1}(0) \hookleftarrow Y_{st} \xrightarrow[/G]{} X$$

$$\tilde{Z} = Z \times T^* \operatorname{Hom}(V', V)$$

this is W, but it is of $\dim V' = \dim V$ and $V' \simeq V$ on the locus that we care about

there is an action of $GL(V')$

$G = GL(V)$ scalars $u \cdot 1_{V} \subset GL(V)$

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$G = GL(V)$

$u' 1_{V'}$ scalars

$u \cdot 1_V \subset GL(V)$

$$\left(\tilde{Z}/G\right)^u \supset Z/G \times \{0\} \sqcup Z^u/G \times \{V' \to V,\ V \not\xrightarrow{0} V'\}$$

$$\begin{tikzcd} V' \arrow[r, bend right, "A"'] & V \arrow[l, bend right, "B"'] \end{tikzcd}$$

$\tilde{Y} = \tilde{\mu}^{-1}(0)$ $\qquad$ $\tilde{\mu} = \mu + AB$

$\tilde{X} = \tilde{Y}_{st}/GL(V)$

any point where $V' \xrightarrow{\sim} V$ is an isomorphism is stable (or the other way around)

we have an action of $GL(V')$

take stable envelope for $u' \subset$ center $GL(V')$

equivariant under $GL(V')$

on the quotient by G

lift to $\tilde{Y}_{st}$ $\longrightarrow$ restrict it to $\tilde{Y}_{iso}$ $\longrightarrow$ descend to $\tilde{Y}_{iso}/GL(V')$

this is the locus where $V' \xrightarrow{\sim} V$

$GL(V')$ acts freely

embed

Nonabelian stable envelope was about extending a class from Y_{st} to Z

$$Z \xhookrightarrow{AB = -\mu} \tilde{Y}_{iso}/GL(V')$$

$$\tilde{Y}_{iso}/GL(V') = \tilde{\mu} = 0$$

$$T^* \mathrm{Hom}\left(\begin{tikzcd} V' \arrow[r, bend right, "A"'] & V \arrow[l, bend right, "B"'] \end{tikzcd}\right)_{iso}/GL(V')$$

the coordinate is $AB \in \mathrm{Hom}(V, V)$.

the coordinate is $AB \in \mathrm{Hom}(V, V')$.

$\tilde{\mu} = 0$

$\tilde{\mu} = \mu + AB$

$$\mathrm{Stab}_{GL(V)}\,\alpha = \text{pullback}\ \underset{GL(V')}{\text{descent w.r.t}}\left(\underset{GL(V)}{\text{lift w.r.t}}\ \mathrm{Stab}_{\tilde{X},u}(\alpha)\right)_{\text{iso}}$$

X is a component of $\tilde{X}^u$

need to check that it is supported on $\mu^{-1}(0)$.

Invariant w.r.t to u and vanishes on $X \subset \tilde{X}$

H is supported on $\tilde{\mu} = 0 = \mu + AB$

therfore $AB = 0$ on the whole u-attracting manifold.

follows from being supported on the attracting manifold. $\square$.

Reference [Bethe eigenfunctions]

$K(\text{stack})$ $\xrightarrow{q}$ $K(X)$

Vertex with descendants

$\overset{!}{\approx}$ ch(nonabelian elliptic stable envelope) for $E = \mathbb{C}^*/q$

① $\mathbb{C}^* \to \mathbb{C}^*/q = E$ induces

$$\mathrm{Spec}\ K_{eq}(X) \to \mathrm{Spec}\ \mathrm{Ell}_{eq}(X)$$

sheaves $\xleftarrow{ch}$ sheaves

more like a hypergeometric function

abbreviation from ch: $\mathbb{C} \xrightarrow{\exp} \mathbb{C}^*$

to have $\overset{!}{=}$ we need

really $ch = (ch_{K \to E})^*$, but this is too long.

① suitable Γ-factors

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② "weak" interpretation

$$(\alpha, \bullet\!\!-\!\!\circ\, \beta) = \text{a function of all variables}$$

in $K_{eq}(\text{stack})$

acts on X

$$X = Z /\!\!/ G \quad , \qquad 1 \to G \to \tilde{G} \to G_{Aut} \to 1.$$

Theorem:

$$\begin{tikzcd}
K_{G_{Aut}}(X)_{z,\,mero} \arrow[r, "ch(\text{elliptic stab for } G)"] \arrow[d, "\Gamma"] \arrow[dr, bend left, "\text{analytic continuation}"] & K_{\tilde{G}}(Z)_{z,\,mero} \arrow[d, "\Gamma' \otimes \hbar^{-\frac{\dim X}{4}}"] \\
K_{G_{Aut}}(X)_{z,\,mero} \arrow[r, "\text{vertex with desc.}"] & K_{\tilde{G}}(Z)_{z,\,mero}.
\end{tikzcd}$$

q-analog of Iritani map

series in z.

$$K_G(X)_{z,\,mero} = \text{meromorphic functions on } \operatorname{Spec} K_G(X) \times \{|q|<1\} \times \operatorname{Pic}(X) \otimes \mathbb{C}^*.$$

z.

[analytic in z in a neighborhood of "0"]

corresponds to stability condition

$$\alpha \bullet\!\!-\!\!\circ\, \beta = (\alpha, \operatorname{Stab}(\beta) \tfrac{\Gamma}{\Gamma})_{stack}$$

$$= \frac{1}{|W|} \int_{|x_i|=1} \prod \frac{dx_i}{2\pi i x_i}\, \alpha(x)\, \operatorname{Stab}(\beta)(x)\, \frac{\Gamma}{\Gamma}$$

fund solution of the q-difference eq.

Last time $\quad \dfrac{\operatorname{stab}_K(\alpha)}{\hbar} \bullet\!\!-\!\!\circ\, \beta = \alpha \longleftarrow\!\!\circ\, \beta$

$\mathcal{L}_\hbar$

Corollary: $$\left(\alpha, \begin{matrix}\text{fund}\\ \text{solution}\end{matrix} \cdot \beta\right) = \frac{1}{|W|}\int_{|x_i|=1} d\,\mathrm{Haar}(x_i)\; f_\alpha(x)\; g_\beta(x)\; \frac{\Gamma}{\Gamma}$$

the K-theoretic stable envelope of α.

the elliptic stable envelope of β

Last time $$\left(\alpha, \begin{matrix}\text{fund}\\ \text{solution}\end{matrix} \cdot \beta\right) = \frac{1}{|W|}\int_{\cancel{|x_i|=1}} d\,\mathrm{Haar}(x_i)\; f_\alpha(x)\; \cancel{g_\beta(x)}\; \frac{\Gamma}{\Gamma}$$

Some contour that depends on stability condition.

Immediately gives analytic continuation and monodromy of <u>vertex functions</u>

interchangeable in the world of q-difference equations

Γ_q have to with maps $\mathbb{A}^1$ or formal disc or,... to X.

$$\mathrm{Maps}\left(\underset{\circlearrowleft\, q}{\mathbb{A}^1} \longrightarrow \underset{\circlearrowleft\, w}{\mathbb{A}^1}\right) = \text{polynomials } 1, t, t^2, t^3, \ldots$$

have weight $w, w/q, w/q^2, \ldots$

weights of the action.

$$\text{character } \vartheta_{\mathrm{Maps}} = \frac{1}{\boxed{(1-w^{-1})}\,(1-qw^{-1})(1-q^2w^{-1})\cdots}$$

$$= \Gamma_q(w^{-1}).$$

where ϑ goes.

much more direct than in cohomology.

if S is a smooth stack $\quad ev_0 : \mathrm{Maps}(\mathbb{A}^1 \to S) \to S$

$$\Omega \qquad - \Gamma(\wedge T^\vee S)$$

If S is a smooth stack

$$ev_{0,*}\,\mathcal{O}_{\mathrm{Maps}(\mathbb{A}^1\to S)} = \Gamma_q(qT^\vee S)$$

extend as multiplicative genus.

with $\hbar$, as in the target of the moment map.

in the diagram: $\Gamma' = \Gamma_q(qT^\vee X - q\mathfrak{g}^\vee + q\mathfrak{g}^*)$

quotient

instead of $-$ because it is good to work with p-fields

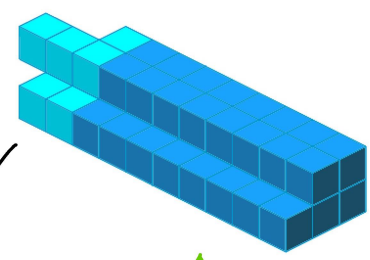
$\Gamma = \Gamma_q($ same but $\cancel{q}\,\mathfrak{g}^*)$

like last time with $\dfrac{1}{\Delta_\hbar}$.

Main ideas of the proof:

In the example of $\mathrm{Hilb}(\mathbb{C}^2, 5)$

quasimaps = PT moduli spaces.



ev_0

ev_∞

1	t_1^{-1}	t_1^{-2}
t_2^{-1}	$t_1^{-1}t_2^{-1}$	

q^2, q^1, q^3

1	t_1^{-1}	t_1^{-2}
t_2^{-1}	$t_1^{-1}t_2^{-1}$	

(1) $$\mathcal{O}_{vir}(\quad) = \frac{ev_\infty(\Gamma_q \text{ expression})}{ev_0(\ \text{—}\ ''\ \text{—}\ )}$$

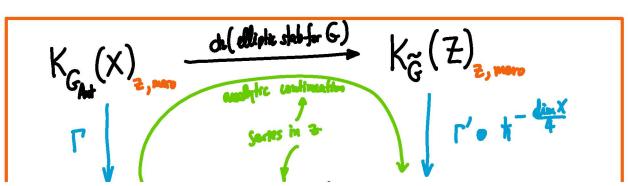
from localiz formula

not symmetrized.

with a tiny difference.

$T_{vir} = H^0(\ldots) - H^1(\ldots) =$

$$= \frac{\text{term at } 0}{1-q^{-1}} - \frac{\text{term } \infty \cdot q^{-1}}{1-q^{-1}}$$



here the blue edges take

$$\begin{tikzcd} K_{G_{Nat}}(X)_{z,\mathrm{nero}} \arrow[r, "\text{vertex with desc.}"] \arrow[d, "\Gamma"'] \arrow[r, bend left, "\text{elliptic cohomology}", "\text{series in } z"'] & K_{\tilde{G}}(Z)_{z,\mathrm{nero}}. \arrow[d, "\Gamma' \circ t^{-\frac{\dim X}{4}}"] \end{tikzcd}$$

← here the blue edges take care of $\mathcal{O}_{vir}$.

$1-q^{\cdot}$ $\quad 1-q$

↳ Γ_q functions.

(2) Let γ be an elliptic stable envelope of some class

$$\frac{ev_0^*(\gamma)}{ev_\infty^*(\gamma)} = \frac{\text{section of some line bundle } (\ldots, q^3 t_1^{-1} t_2^{-1}, \ldots)}{\text{section of some line bundle } (\ldots, t_1^{-1} t_2^{-1}, \ldots)}.$$

z^{-3}

$$= z^{-\text{degree}} \quad \text{constant in } z.$$

↳ turns $\mathcal{O}_{vir}$ into $\hat{\mathcal{O}}_{vir} \otimes \hbar^{-\frac{1}{4}\dim X}$.

(3) Using 1 and 2 we can move

it remains to forget that the general pt here needs to be stable.

$$\bullet\!\!-\!\!\circ \;=\; QM(X) \underset{\text{stable}}{\subset} \mathrm{Maps}(\mathbb{A}^1, \text{stack}).$$

will give the Γ_q functions.

need to extend the integration from the stable locus to the whole thing.

⇗ the hardest part [Inductive ..., II, 3.4]