Elliptic stable envelopes for $\mathcal{N}=2$ applications

$X = \mathcal{Z} \cap G$
$= \mu^{-1}(0) \cap G$
$= \mu^{-1}(0)_{st} / G$

$\text{st} = \infty$
no finite stabilizers

Like the abelian stable envelope: 0) nonabelian stable envelope solves an interpolation problem

$Z_{st} \hookrightarrow Z$

$\text{Ell}_q(Z_{st}) \rightarrow \text{Ell}_q(Z)$

also must vanish under
also must vanish under
\[ Z \setminus Y \xrightarrow{\text{open}} Z \]
\[ \text{Ell}(Z \setminus Y) \xrightarrow{\text{closed}} \text{Ell}(Z) \]

"wheel conditions"

for \( \text{Kill}(\mathfrak{g}, n) \), \( Z = \text{Hom}(V, V) \otimes (t_1 + t_2) \otimes V \otimes V^* \cdot t_1 t_2 \)

\[ \text{GL}(n) \ni \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \]
\[ \{ x_i, x_j, x_k \} = \{ c, ct_1, ct_2 \} \quad \text{or} \quad \{ c, ct_2, ct_1 \} \]

\[ \square \quad \text{or} \quad \square \]

\( \square \)(or) \( \square \)

\( \text{contents.} \)

\( \text{it may be solved inductively,} \)
\( \text{using a stratification of the unstable locus discussed last time.} \)

\( \text{takes values in } J_0 = \bigoplus (T^{1/2}Z) \otimes \text{degree zero} \)
\( \text{or, more specifically } \)
\[ J = \text{Ker} (\text{restriction to } Z \setminus Y) \]

\( \text{wheel conditions} \)

main source: \( G \)-equivariant line bundles on \( Z \)
especially \( L = O_Z \otimes \text{character } \chi \) of \( G \)
\( U(\chi, z) \) a section has the form
\[ \frac{\theta(\chi(z))}{\theta(\chi(x)) \theta(z)} \]
\[ \text{dual \text{\'e}tale \text{\'e}tale variable} \]
\( \text{in the Grass.} \)
\( \frac{\text{\text{\'e}tale}(x_1 z \ldots)}{\text{\text{\'e}tale}(x_1 \ldots) \text{\text{\'e}tale}(z)} \)
cancels with poles.

\( \text{If } G = \prod_{i} \text{GL}(V_i) \text{ then these GIT stable} \)
\( \text{envelopes reduce to abelian ones} \) later
(4) If $G = \prod_{i \in I} GL(V_i)$, then their GIT stable envelopes reduce to abelian ones \( \leftarrow \text{later} \)

Applications

1. [next time, I hope] \( \longrightarrow \) elliptic stable envelopes \( \frac{\Gamma_q}{\Gamma_q} \)

\[ K(\text{stack}) \quad K(\mathcal{X}) \]

2. today we relate \( \longrightarrow \) to the $K$-theoretic $\text{Stab}$ for the stack

Let \( \alpha \in K_{\text{eq}}(\mathcal{X}) \), \( \text{stab}(\alpha) \in K_{\text{eq} \times G}(\mathcal{Z}) \) supported on \( \mu = 0 \).

Thm [Zhele paper with Mina]

\[ \frac{\text{stab}(\alpha)}{\Delta_{\mu}} \quad \longrightarrow \quad \alpha \quad \longmapsto \]

for \( \text{GL}(n) \)

\[ \prod_{i,j} (1 - t x_i / x_j) \]

\[ \Delta_{\mu} \] weight of $\mu$

\[ \text{for GIT} \]

\[ \prod_{i,j} (1 - t x_i / x_j) \]

\[ \frac{d^k}{dt^k} \mu(f(t)) = 0 \]

\( k = 1, 2, \ldots \)

Corollary

\[ \text{Stab}(\alpha) \]

\[ \frac{1}{\Delta_{\mu}} \] comaps with $\Delta_{\mu}$ in localization.

Relative $\alpha$

\[ \mathbb{P}^1 \rightarrow \mathbb{Z} \]

$s + t \mu(f(t)) \equiv 0$.

$\mu(\alpha) = \text{evaluation at 0}$. Compare & contrast with Smirnov's formula:

\[ \text{no zero and no pole in } \text{Stab}(\alpha) \]

\[ K \text{-theoretic} \]

\[ \text{here we study maps } f: \mathbb{P}^1 \rightarrow \mathbb{Z} \]
Proof: Bubble it off

\[ \frac{\text{Stab}(d)}{\Delta^h} \]

apply the theorem

\[ \int \frac{dx_i}{x_i} \frac{\text{Stab}(d)(x)}{\prod (1-t x_i q) \prod \gamma_q} \]

fundamental solution of our q-difference equations

\[ \left( \text{Moduli spaces of } x_0 \right)^G = \text{GIT quotients by } G \]

\[ \mathcal{F} = G \text{- invariant sheaf on } \mathcal{L} \]

\[ \chi(\mathcal{L}/G, \mathcal{F}^G) = \chi(\mathcal{L}, \mathcal{F})^G \]

max compact tors in G

\[ \mathcal{I} \]

by Weyl integration formula

have to do steepest descent

\[ \left( \text{in } m \right)^G \]

m \to 0.

\[ \chi \]

contour that depends on \( \chi \).
\[ \text{(Value at } m=0) \quad \chi \left( \frac{z}{G}, \left( F \circ f^m \right) G \right)_{m \to 0} \]

[Appendix to AFO]

Conclusion:
\[ \langle \alpha | \text{ fundamental solution of } q\text{-diff. } | \beta \rangle = \int \text{stab}(x) \frac{\Gamma_q}{\Gamma_q} \frac{dx}{2\pi i x}; \]

\[ \text{cycle(} \beta \text{)} \to \exp \left( \frac{W(x)}{\ln q} \right) \]

in this formula, we can take \( q \to 1 \).

\[ \Psi(q) = M(z, q)^\dagger \Psi(q) \]

\( q \to 1, \quad \Psi \sim e^{z} \psi_1 \)

\[ \text{eigenvector of } M(z, 1) \]

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\[ \text{Stab}(x) = \text{off-shell Bethe eigenfunction} \]

\[ z = \text{parameter (similar to quasiperiodic b.c. in spin chains)} \]

\[ \chi_1 = \text{classical multiplication in } K_q(x) \]

\[ \text{fixed pts.} \]

\[ \text{Spectrum } = \text{Spec } K_q(x) \]

\[ z=0 \]

\[ p \in \text{Torus, } \sigma_p \in K_q(x) \text{ is an eigenvector} \]

\[ z \to 0 \]

\[ \text{Bethe equations} \]
\( p \in X, \quad O_p \in K_0(X) \) is an eigenvector
\[ F \cdot O_p = F \big|_p \cdot O_p. \]

going back to theorem [A.-0.]
\[ \frac{\text{Stab}(\alpha)}{\Delta_h} \rightarrow \alpha \rightarrow \]

by localization
\[ \frac{\text{Stab}(\alpha)}{\Delta_h} \rightarrow \alpha \rightarrow 1 \rightarrow 0 \text{ as } q \rightarrow \infty \]

edge term in loc. formula
\[ 1 \rightarrow \text{Glue} \rightarrow 0 \rightarrow 1 \rightarrow \text{Glue} \rightarrow \]

proper, so
Lament pity in \( q \)
here we have to show that

\[ \alpha \rightarrow 0 \quad \text{as } q \rightarrow 0 \]

boundary \( \rightarrow 0 \quad \text{as } q \rightarrow \infty \)

\[ \text{tropical rigidity proof} \]

Preview of:

1. If \( G = T GTL_i \) then these GIT stable envelopes reduce to abelian ones \(-\) later

for \( X = \text{Hilb}(\mathbb{C}^2, n) \)
\[ \alpha \rightarrow \text{Stab} \rightarrow \text{symmetric function of } x_1, x_2, \ldots, x_n. \]
\[ \bigotimes_{i=1}^{n} \text{Fock}(x_i) \]

we will find it as a matrix element of \( R \) matrix in
we will find it as a matrix element of $R$ matrix in

$$\text{Hilb}(\mathbb{A}^2, n) \times \text{sheaves of rank } n$$

with \((x_1, x_n)\) playing the role

\((q_1, q_n) \in \text{GL}(W)\).

Fock representation of elliptic $q-$... quantum group.

\(\{1, t_1, t_2, t_3, t_4\}\)

\(\{x_1, x_2, x_3, x_4\}\).

Each is at the origin $x_i y_j = 0.$

Exercise: recast $T^* Gr$ in the same form.

$B(x_1) B(x_2) \ldots |\text{vac}\rangle.$