Lecture 31, 12/15/2020

Tuesday, December 8, 2020 9:31 PM

Enumerative geometry & geometric representation theory

Start time Moscow ~~17:30~~ 18:30
New York 10:30

Elliptic stable envelopes for /// & applications

algebraic symplectic reductions

$$X = Z /\!\!/\!\!/ G$$

smooth & symplectic

$$= \mu^{-1}(0) /\!\!/ G$$

$$= \mu^{-1}(0)_{st} / G$$

st = ss

no finite stabilisers

$$\begin{tikzcd}
 & Z & \\
\Theta(T^{1/2}Z)\otimes\ldots\ Z_{st} \arrow[ur, "!"] & & Y = \mu^{-1}(0) \arrow[ul, "\text{extend, so that it is supported on } Y"'] \\
 & Y_{st} \arrow[ul, "\text{push-forward}"] \arrow[ur] \arrow[d, "/G"] & \\
\Theta(T^{1/2}X)\otimes\ldots\ X & &
\end{tikzcd}$$

$\otimes\ \Theta(\mathfrak{h}^{-1}\mathfrak{g}^*)$, μ

problematic, since Y is not smooth

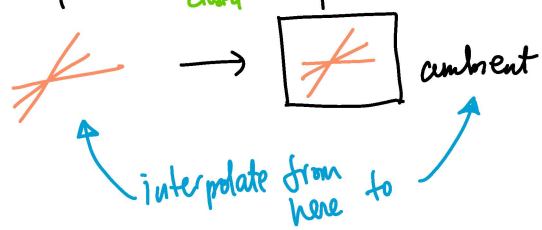
pull back

class α

Like the abelian stable envelope: (1) nonabelian stable envelope solves an <u>interpolation problem</u>

$$Z_{st} \overset{\text{open}}{\hookrightarrow} Z$$

$$\mathrm{Ell}_{eq}(Z_{st}) \xrightarrow[\text{closed}]{} \mathrm{Ell}_{eq}(Z)$$



also must vanish under

open

also must vanish under

$$Z \setminus Y \overset{\text{open}}{\hookrightarrow} Z$$

$$\mathrm{Ell}(Z \setminus Y) \xrightarrow[\text{closed}]{} \mathrm{Ell}(Z)$$

"wheel conditions"

for $\mathrm{Hilb}(\mathbb{C}^2, n)$, $\quad Z = \mathrm{Hom}(V,V) \otimes (t_1 + t_2) \oplus V \oplus V^* \cdot t_1 t_2$

$$GL(n) \ni \begin{pmatrix} x_1 & & \\ & x_1 & \\ & & \ddots \\ & & & x_n \end{pmatrix} \qquad \{x_i, x_j, x_k\} = \{c, ct_1, ct_1t_2\}$$

or $\{c, ct_2, ct_1t_2\}$

contents.

(2) it may be solved inductively, using a stratification of the unstable locus discussed last time.

(3) takes values in $\mathcal{S}_0 = \Theta(T^{1/2} Z) \otimes$ degree zero

or, more specifically $\mathcal{S} = \mathrm{Ker}(\text{restriction to } Z \setminus Y)$

wheel conditions

main source are G-equivariant line bundles on Z

especially $\mathcal{L} = \mathcal{O}_Z \otimes$ character χ of G

$\mathcal{L} = \mathcal{U}(\chi, z)$ a section has the form "$\dfrac{\theta(\chi(x) z)}{\theta(\chi(x))\,\theta(z)}$" $\longrightarrow$ $\dfrac{\prod \vartheta(x_i z \ldots)}{\cancel{\prod \vartheta(x_i \ldots)}\, \prod \vartheta(z)}$

dual Kähler variable

in the Grass.

cancels with polaris.

(4) If $G = \prod_i GL(V_i)$ then these GIT stable envelopes reduce to abelian ones $\leftarrow$ later

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<u>Applications</u>

(1) [next time, I hope] $\bullet\!\!-\!\!\circ$ = elliptic stable envelopes $\cdot \frac{\Gamma_q}{\Gamma_q}$

$K(\text{Stack})$ $\quad K(X)$

(2) today we relate $\bullet\!\!\to$ to the K-theoretic Stab for the stack. $q \to 0$ $\quad \frac{\ln z}{\ln q} \to \dots$

Let $\alpha \in K_{eq}(X)$, $\quad \mathrm{stab}(\alpha) \in K_{eq \times G}(Z)$ supported on $\mu = 0$.

<u>Thm</u> [Bethe paper with Mina] $\quad \frac{\mathrm{stab}(\alpha)}{\Delta_\hbar} \longrightarrow = \alpha \longleftrightarrow$

weight of μ

compare & contrast with Smirnov's formula no z and no q in $\mathrm{Stab}(\alpha)$ ← K-theoretic

for $GL(n)$ $\quad \prod_{i,j} (1 - \hbar x_i / x_j)$

$\mathrm{Stab}(\alpha) \overset{\prime}{\longrightarrow}$

$$\frac{d^k}{dt^k} \mu(f(t)) = 0 \qquad k = 1, 2, \dots$$

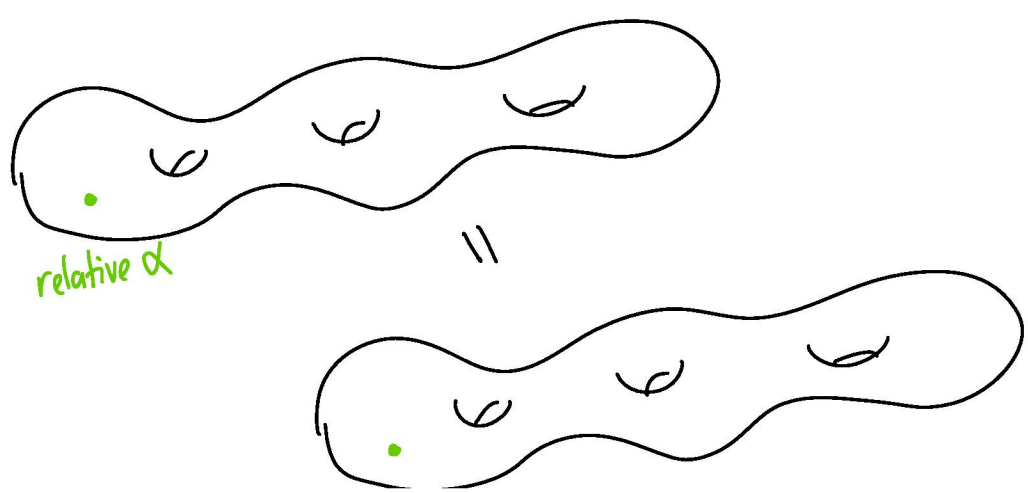
takes care of $\mu(f(0)) = 0$

here we study maps $f: \mathbb{P}^1 \to Z$ s.t. $\mu(f(t)) \equiv 0$.

$f(0)$ = evaluation at 0.

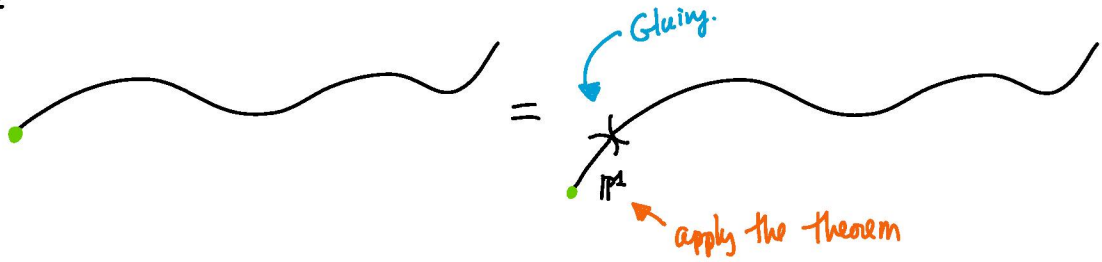
$\frac{1}{\Delta_\hbar}$ cancels with $\Delta_\hbar$ in localization formula

<u>Corollary</u>



descendent insertion $\frac{\mathrm{Stab}(\alpha)}{\Delta_{\hbar}}$ or with'

<u>Proof:</u> Bubble it off



$\square$

<u>Corollary 12</u>

$$\alpha \longleftarrow \beta \;=\; \frac{\mathrm{Stab}(\alpha)}{\Delta_{\hbar}} \bullet\!\!-\!\!\circ$$

fundamental solution of our q-difference equations

$$\int_{\gamma_\beta} \prod \frac{dx_i}{x_i} \frac{\mathrm{Stab}(\alpha)(x)}{\prod(1-\hbar x_i/x_j)} \prod \frac{\Gamma_q}{\Gamma_q}$$

! $\left(\text{Moduli spaces of } \bullet\!\!-\!\!\circ\right)^{q} = \text{GIT quotients by } G$

$\mathcal{F} = G$-equivariant sheaf on $\mathcal{Z}$

$$\chi(\mathcal{Z}/G, \mathcal{F}^G) = \chi(\mathcal{Z}, \mathcal{F})^G$$

$\int$ by Weyl integration formula

max compact torus in G

$$\chi(\mathcal{Z}/\!\!/_{\chi} G, \mathcal{F}^G) = \int \text{—}''\text{—}$$

contour that depends on χ.

have to do steepest descent

$\chi(\mathcal{Z}, (\mathcal{F} \otimes \chi^m)^G)$ $\quad m \gg 0.$

$$\left(\text{value at } m=0 \quad \chi\left(\mathcal{Z}/G, (\mathcal{F}\otimes\mathcal{L}^m)^G\right)\right) \qquad m \gg 0.$$

[appendix to AFO]

Conclusion:

$$\left\langle \alpha \,\middle|\, \begin{matrix}\text{fundamental}\\ \text{solution of q-diff.}\\ \text{equations}\end{matrix} \,\middle|\, \beta \right\rangle = \int_{\mathrm{cycle}(\beta)} \mathrm{stab}(\alpha)\, \frac{\Gamma_q}{\Gamma_q} \prod \frac{dx_i}{2\pi i x_i}$$

$\frac{\Gamma_q}{\Gamma_q} \to \exp\left(\frac{W(x,\dots)}{\ln q}\right)$

K-theoretic in particular has q

in this formula, we can take $q \to 1$.

$$\Psi(qz) = M(z,q)\Psi(z)$$

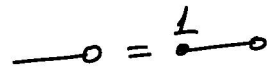
$$q\to 1, \quad \Psi \sim e^{\frac{1}{\ln q}\int \lambda_i}\, \psi_i$$

eigenvalue

eigenvector of $M(z,1)$

saddle point $\frac{\partial}{\partial x_i} W = 0.$ = Bethe equations

Nekrasov–Shatashvili



$\mathrm{Stab}(\alpha)$ = off-shell Bethe eigenfunction

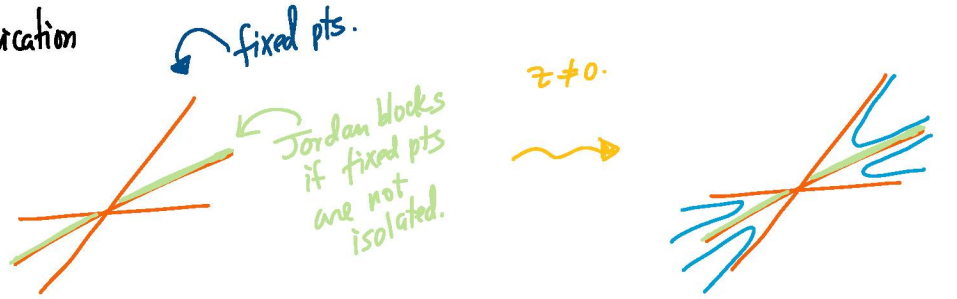
= der Ansatz

z = parameter (similar to quasiperiodic b.c. in spin chains)

$z=0$

QIS = classical multiplication in $K_{eq}(X)$

Spectrum = Spec $K_{eq}(X)$



$p \in X^{\text{torus}}$, $\mathcal{O}_p \in K_{eq}(X)$ is an eigenvector

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$$\mathcal{F} \otimes \mathcal{O}_p = \mathcal{F}|_p \otimes \mathcal{O}_p.$$

Bethe equations

$$\mathrm{Stab}(\cdots)\Big|_{\text{Bethe eq.}} = \text{eigenvectors}$$

Going back to Theorem [A.–O.]

Tube = slue from earlier lectures

does not depend on q

$$\frac{\mathrm{Stab}(\alpha)}{\Delta_\hbar} \longrightarrow \; = \; \alpha \longleftrightarrow$$

K-theoretic

$\Uparrow$

slope + polarization need to be fixed correctly

by localization

edge term in loc. formula

$$\underbrace{\frac{\mathrm{Stab}(\alpha)}{\Delta_\hbar}}\; \bullet\!\!-\!\!\circ \; \circ\!\!\longrightarrow \; =$$

proper, so Laurent poly in q

here we have to show that

$$\underbrace{\frac{1}{1-q\psi_\infty}}$$

as in the slue = tube

$\to$ Glue as $q \to 0$

$\to 1$ as $q \to \infty$.

$$\boxed{\begin{array}{lll} \alpha & \text{as} & q \to 0 \\ \text{bounded} & \text{as} & q \to \infty \end{array}}$$

typical rigidity proof

$\square$

<u>Preview</u>: of

$$\boxed{\text{④ If } G = \prod_i GL(V_i) \text{ then these GIT stable envelopes } \underline{\text{reduce}} \text{ to abelian ones} \leftarrow \text{later}}$$

for $X = \mathrm{Hilb}(\mathbb{C}^2, n)$

$$\alpha \xrightarrow{\mathrm{Stab}} \text{symmetric function of } x_1, x_2, \ldots, x_n.$$

$$\bigotimes_{i=1}^{n} \mathrm{Fock}(x_i)$$

we will find it as a matrix element of R matrix in

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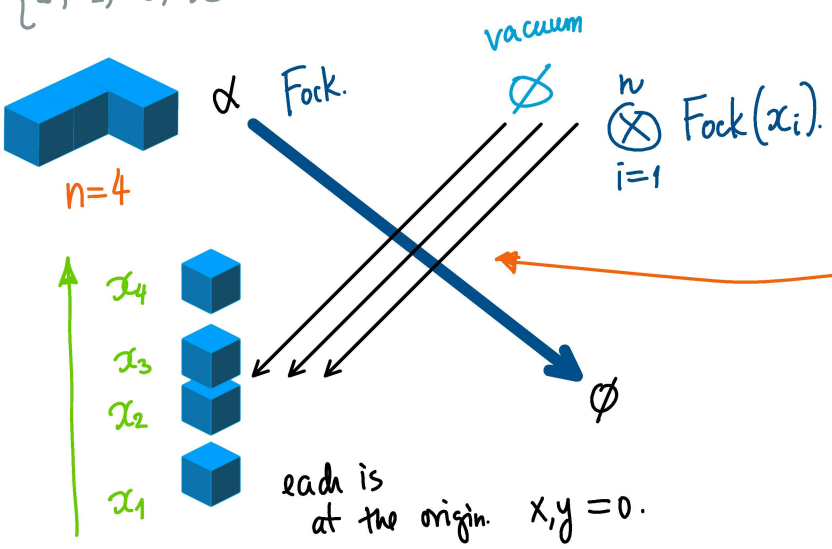
$\mathrm{Hilb}(\mathbb{C}^2, n) \times$ sheafs of rank n

with $\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}$ playing the role $\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \in GL(W)$.

Fock representation of elliptic / q-... / ... quantum group

$\{1, t_1, t_2, t_2^2\}$



$\{x_1, x_2, x_3, x_4\}$.

this matrix element of R-matrix <u>is</u> the nonabelian stable envelope.

<u>exersize</u>: recast T^*Gr in the same form.

$B(x_1) B(x_2) \cdots |\mathrm{vac}\rangle$.