Lecture 27, 11/17/2020

Tuesday, November 17, 2020 11:01 AM

Enumerative geometry & geometric representation theory

Start time Moscow ~~17:30~~ 18:30

New York 10:30

$\circlearrowleft q \longleftarrow \circ \;\in\; \mathrm{End}(K_{eq}(X))_{loc}$ depends on

equivariant variables $T \subset \mathrm{Aut}(X)$ and Kähler variables $z \in \mathrm{Pic}(X) \otimes \mathbb{C}^\times$ $\to z^{\deg}$

fundamental solution for a flat q-difference connection in both.

$$1 \to A \to T \xrightarrow{\hbar} \mathbb{C}^* \to 1$$

weight of ω_X

only consider shifts in these variables, do not attempt $\hbar \to \hbar q$ possible, but very complex like $t \mapsto tq$ in Macdonald $\mathcal{U}_\hbar \mapsto \mathcal{U}_{q\hbar}$

A vs. $Z = \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}^*$

(?) Where are the singularities, mod q = ?

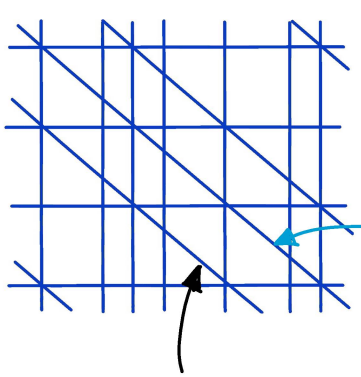
$$\Psi(qa) = S(a,q)\,\Psi(a)$$

Def. Singularities $= \{ S \notin GL(n) \} \cup \{ a = 0, \infty \}$

singularities in $A/q^{\cdots} = \mathrm{Ell}_A(pt)$

$$\begin{tikzcd} \mathrm{Ell}_A(X) \arrow[d] & H^*(X^a) \arrow[l, "\log"'] \arrow[d] \\ \mathrm{Ell}_A(pt) & a \arrow[l] \end{tikzcd}$$

$\mathrm{Lie}(A)$



periodic arrangement of hyperplanes (shifts of $\mathrm{Ell}_{A'}(pt)$, $A' \subset A$ (codim = 1))

equivariant roots of X.

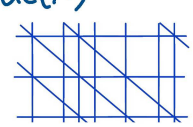
Consider the normal bundle $N_{X/X^A} \circlearrowleft A$

$\{w_i\}$ = weights of A in N_{X/X^A}.

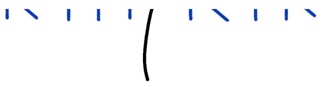
$\cap$

characters(A)

$\mathrm{Lie}(A)$



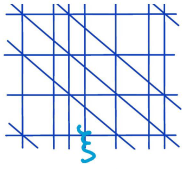
$\langle \xi, w_i \rangle \in \mathbb{Z}$.



these are the points a such that $X^a \neq X^A$

(shifts of $\mathrm{Ell}_{A'}(pt)$

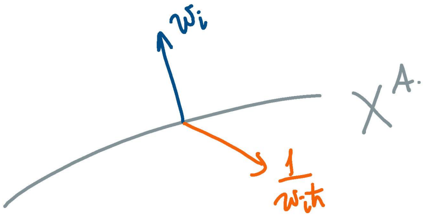
$A' \subset A$ codim=1)

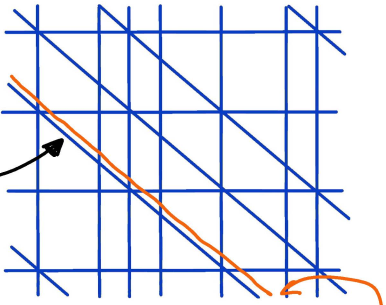


$\langle \xi, w_i \rangle \in \mathbb{Z}$.

$\hookrightarrow a = e^{2\pi i \xi}$

More precisely,





Vertex function for $\mathbb{P}^1$

$$\sum_d z^d \frac{(\hbar)_d (\hbar a)_d}{(q)_d (qa)_d},$$

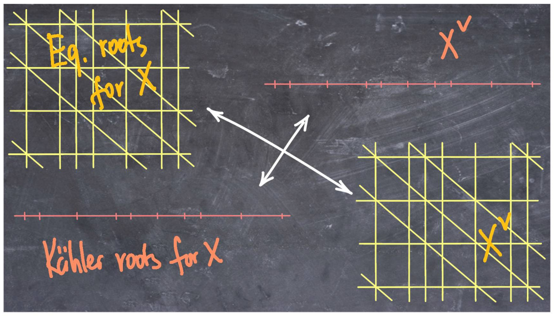
rational if $\hbar = q^n$.

$\| \; 1 \mod (\hbar - 1)$

annihilate at $\hbar = 1$. or $\hbar = q^n$

( )

$\Longrightarrow$ on the Kähler side there will be Kähler roots $\subset$ characters Z
$\| $
$H_2(X, \mathbb{Z})$



Recall,

Among all difference equations in equivariant variables, we want to find qKZ

$$S_\sigma = \xleftrightarrow{\sigma}$$

we count with $z^{\deg}$, as usual

formal power series in z.

$$\Psi(qa_1, a_2) = \overbrace{(Z \otimes 1)\, R_{12}(a_1/a_2)}^{S_\sigma}\, \Psi$$

no q here

appears only outside

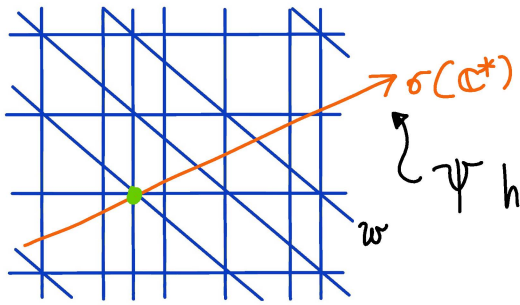
z appears only outside as a monomial

no q here

in [Frenkel-Reshetikhin, ...]

$q \approx \cancel{\hbar}$ level

for us, q is free !



Ψ has only singularities at q^n, $n \in \mathbb{Z}$

typically, not the case

$\langle w, \sigma \rangle = n \rightsquigarrow$ get singularities at $\sqrt[n]{q}$

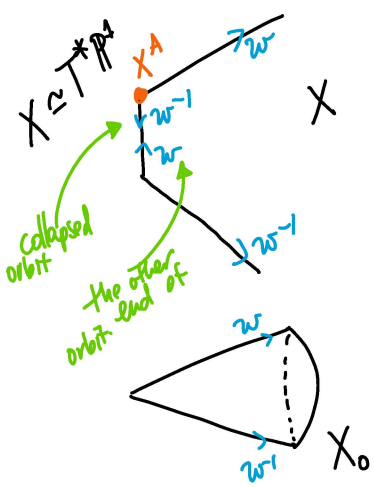
$w(\sigma(b))\, q^m = b^n q^m$

Singularities at $q^{\frac{1}{n}\mathbb{Z}}$.

Has to be the case that $\boxed{|\langle \sigma, \text{root} \rangle| \leq 1.}$

definition, as in Lie theory

Means σ has to be minuscule



σ acts with weights $\pm 1, 0$

$X_0 \subset$ vector space

functions here generate functions on X_0
$\parallel$
functions on X

Def' σ is minuscule if $\mathbb{C}[X]$ is generated by functions of weight $= 0, \pm 1$ w.r.t to σ

FFT of invariant theory

Exercise.

$$\sigma(b) = \begin{pmatrix} b & & & & & \\ & b & & & & \\ & & b & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \in GL(W) \subset \mathrm{Aut}(\text{Nakajima}, \omega)$$

is minuscule.

framing

$v(\infty) = \begin{pmatrix} & & -1 \\ & & \\ 1 & & \end{pmatrix}$

minuscule for $GL(n)$.

framing

means S_σ has a chance to be qKZ

<u>Back to</u>.

$S_\sigma =$ 

we count with $z^{\deg}$, as usual

formal power series in z.

$$\Psi(qa_1, a_2) = (z \otimes 1)\, R_{12}(a_1/a_2)\, \Psi$$

no q here

z appears only outside as a <u>monomial</u>

only "constant" quasimaps contribute

locally constant on components.

$$(z \otimes 1) = z^{\text{degree (constant to } X^\sigma)}.$$

All other contributions will vanish $\longleftarrow$ by <u>rigidity</u>

minuscule is important

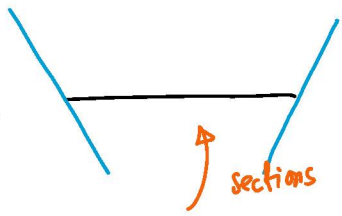
properness + bound the weights

the result is a Laurent poly.

the Newton polygon of that Laurent poly contains no lattice points.

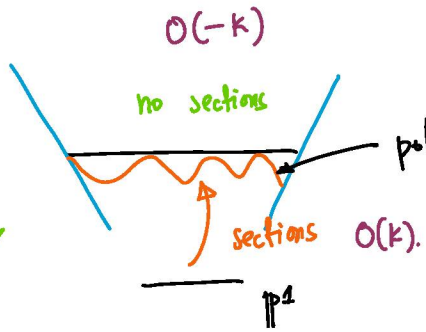
<u>Properness</u>

Silly example: $X = \mathbb{C}^2$ $\quad \sigma(b) = \begin{pmatrix} b^k & 0 \\ 0 & b^{-k} \end{pmatrix} \subset \mathrm{Aut}(X, \omega)$.

$\tilde{X} =$ 

sections

$= \mathcal{O}(k) \oplus \mathcal{O}(-k)$. $\quad k > 0$.



polynomials of degree k.

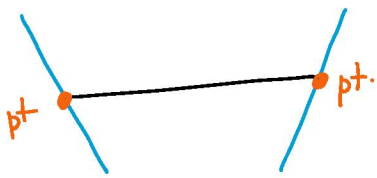
bad for properness.

can restrict the values of the section at 0 and ∞

class on X $\longleftrightarrow$ class on X

$k > 1$.

First approximation:



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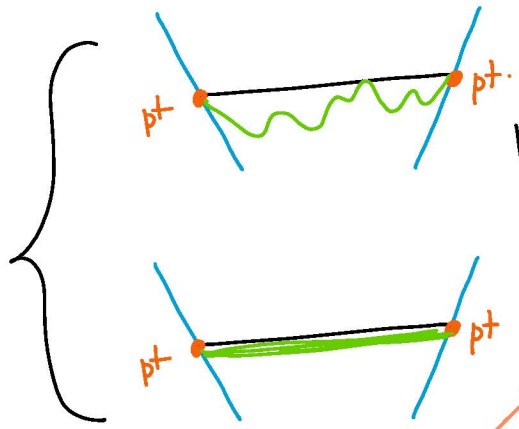


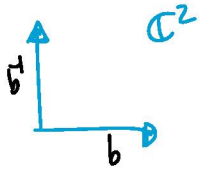
$H^0(\mathcal{O}(k-[0]-[\infty]))$

$k = 1$

proper !

minuscule !!!

Not great when we will start counting weights

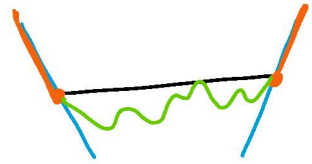


$\mathcal{O}_{pt} = (1-b^{-1})(1-b)$ Newton polygon( ) $= [-1, 1]$

We can do better

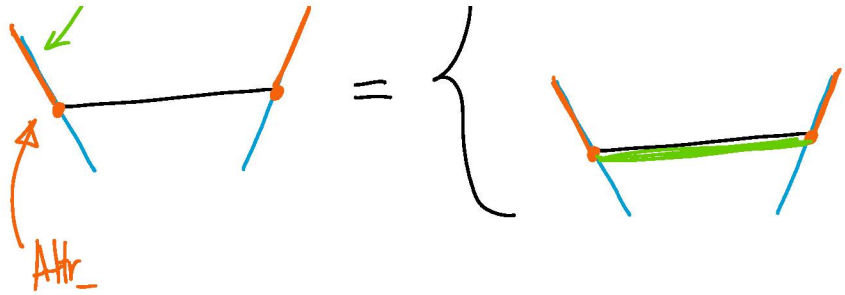
The section can't go in this direction anyway

$=$



$k > 1$.

$H^0(\mathcal{O}(k-[0]-[\infty]))$



$k=1$

proper !

minuscule.

for general X

$$X \xrightarrow{\text{proper}} X_0 \xrightarrow{\text{proper}} \text{Vector space with weights } 0, \pm 1.$$

$$\sigma\text{-twisted } QM(X) \xrightarrow{\text{proper}} \text{sections of } \mathcal{O}(1)^{\cdots} \oplus \mathcal{O}(-1)^{\cdots} \oplus \mathcal{O}.$$

σ-twisted QM(Vector space).

properness here is like in the silly example.

for constant maps get $\pm z^{\deg}$

Thm

$$\begin{tikzcd} K(X^\sigma) \arrow[r] \arrow[d, "\mathrm{Stab}_-"'] & K(X^\sigma) \\ K(X) \arrow[r, "\sigma"] & K(X) \arrow[u, "\mathrm{Stab}_-^{\text{Transpose}}"'] \end{tikzcd}$$

① is proper

② vanishes for nonconstant maps

actual work [pcmi] Section 10.2

repelling directions

K-theoretic stable envelope depends on slope

Kähler variables in Elliptic theory.

chosen with care

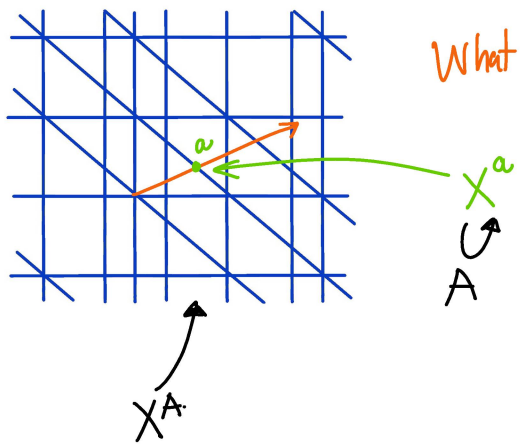
$$\mathrm{Stab}_-^{\text{transpose}} = (\mathrm{Stab}_+)^{-1}.$$

R- matrix is $\mathrm{Stab}_-^{-1} \circ \mathrm{Stab}_+$ we get a digram

$$\begin{tikzcd} K(X^\sigma) \arrow[r, "qKZ"] \arrow[d, "\mathrm{Stab}_+"'] & K(X^\sigma) \arrow[d, "\mathrm{Stab}_+ = ((\mathrm{Stab}_-)^T)^{-1}"] \\ K(X) \arrow[r, "S_\sigma"] & K(X) \end{tikzcd}$$

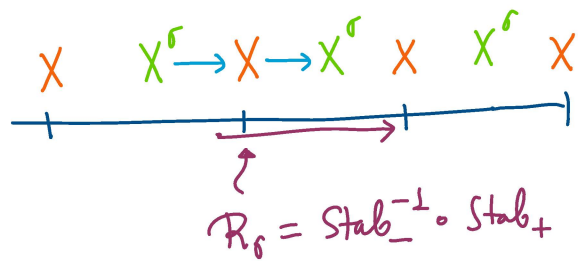
from this, one compute difference equations in Kähler variables $\Rightarrow$ next time.

?



What about nonminuscule shifts σ ?

in qKZ



there is a natural conjecture

$$S_\sigma = \overrightarrow{\prod} \text{ R-matrices for } X^a$$

all root hyperplanes crossed by σ

also a Nakajima quiver variety

Good progress by Y. Kononov and A. Smirnov