$\sigma : \mathbb{C}^* \to T \subset \text{Aut}(X)$

vs.

fixed by $q \in \mathbb{C}^* \subset \mathbb{P}^1$

from the difference in pure edge

t $\mapsto \sigma(q)t$ \quad $t \in T$

$\delta : \mathbb{C}^* \to \text{Aut}(Y)$, e.g. $Y = \mathbb{P}^1$

gives

$Y \leftarrow Y' \downarrow \mathbb{P}^1$

degree of the "constant map" in example, get $z^{-4}$

$[S_0] - [S_{oo}] = k [\text{Fiber } \mathbb{P}']$

points of $Y^\sigma$ give "constant sections"

$\mathcal{O}(k)$ normal bundle

Measure the degrees of curves by pairing them with line bundles $[\text{curves}] \in \text{Pic}^V$

degree in $\mathbb{P}^1$ direction = 1.

$0 \rightarrow \mathbb{Z} \rightarrow \text{Pic}(Y) \rightarrow \text{Pic}(Y) \rightarrow 0$

$\sigma : \mathbb{C}^* \rightarrow T \subset \text{Aut}(X)$

if $q \in \mathbb{C}^* \subset \mathbb{P}^1$
\[
\langle L, \text{constant section } y \in Y^r \rangle = \langle \sigma, L \mid y \rangle \\
\text{deg}_2(\text{constant section}) \quad \text{codimension} \quad \text{character} \\
\text{in example} \quad \langle O(1) \mid_0 \rangle = q \cdot q^l \\
\langle O(1) \mid_\infty \rangle = 1 \cdot q^l \\
\langle \sigma, \cdot \rangle \text{ these differ by } k \\
-4 = \langle (t_1, t_2), O(1) \mid_y \rangle = \frac{t_1^4}{t_1^2} \\
(q, 1) \\
\]

- by inspection 
- by degeneration 
- \( q \)-difference operator 
- degree of the constant map

This means: \( \Psi = \text{the operator } \epsilon \rightarrow \)
= \( \Psi(z, t, q) \)

\[
\Psi(z, t, q) = \Psi(z, \sigma(q)t, q) \frac{\Gamma}{\Gamma} z \\
\]

\( S_q(z, t, q) \)

\( \sigma \) this is a \( q \)-difference eq. in equivariant variables
This is a q-difference eq. in equivariant variables

\[ S_q(\omega_1, q) \]

\[ \Gamma_q(\omega_1) \Gamma_q(\omega_2) \approx \Theta(\omega) \]

Very clearly connected with

"attractive" line bundles \[ J = \bigwedge (T_{\mathcal{X}} \cdot \mathcal{U}) \]

in the discussion of elliptic stable envelopes

\[ \prod \frac{\Theta(z_i \xi_i)}{\Theta(z_i) \Theta(\xi_i)} = \bigwedge (\sum (z_i - 1)(\xi_i - 1)) \]

\[ \zeta \text{ a basis of Pic} \]

A q-difference equation in Kahler variables
by localization or by the structure of \( K(\mathbb{P}^1) = \)

we have:

\[ c_1(L_0) - c_1(L_{\infty}) = \deg L \cdot c_1(T_0 \mathbb{P}^1) \]

the weight

\[ L_0 \otimes L_{\infty}^{-1} = (T_0 \mathbb{P}^1)^{\deg L} \]

\( F \) is any sheaf on \( \mathbb{P}^1 \), \( \det F \) is a line bundle, \( \deg F = \deg \det F \)

\[ q = \frac{\deg F}{\det F_0} \]

\[ \frac{\det F_0}{\det F_0} \]

The same formula for the degree of the constant map
\[ \chi(\mathcal{M}, \mathbb{Z}, \deg \hat{\Theta}_{\text{vir}} \otimes \text{insertions}) \mid_{\tau \to q^2} = \chi(\mathcal{M}, \mathbb{Z}, \deg \hat{\Theta}_{\text{vir}} \otimes \text{insertions} \otimes \det(F_0) \otimes \det(F_1)^{-1}) \]

if \( \mathcal{M} = \{ \star \overset{0}{\to} \alpha \} \) then

\[ \star \overset{0}{\to} \alpha \mid_{\tau \to q^2} = \star \overset{0}{\to} \alpha \otimes L^{-1} \]

where \( * = \det(F \otimes \Theta_0) \)

by degeneration

Recall \( \Psi = \star \overset{0}{\to} \alpha \)

\[ \Psi(q^2, t, q) \otimes L = M_2 \Psi(t, t, q) \]

\[ M_2 \Psi(t, t, q) \]

\[ \star \overset{0}{\to} \alpha \otimes L^{-1} \]

\[ \tau \in \text{Pic}(X) \otimes \mathbb{C}^* \]

Pic = cocharacters

\[ \text{q-difference equation in Kähler variables} \]

Main feature: commutes with the

\[ \text{in equiv. variables} \]
Goal: describe this flat q-difference connection in quantum group terms

Plan: start with equations for certain special equivariant variables

Motivation: $K(T^*Gr(k,n)) = \text{integral form of}$, weight $k$ subspace in

$GL(n) = GL(W)$ changes of framing

$\bigotimes (\mathbb{C}^2(a_1) \otimes \mathbb{C}^2(a_2) \otimes \ldots \otimes \mathbb{C}^2(a_n))$ as $U_q(\widehat{sl}_n)$ module where

$qKZ$ connection moves as around.

Clearly this must be the connection we want, no?

More generally:

$$Q = \begin{pmatrix} V_1 & V_2 & \ldots \\ \vdots & \vdots & \ddots \\ W_1 & W_2 & \ldots \end{pmatrix}$$

$GL(W) = \bigotimes GL(W_i)$. $w_j$ i-th fund eval repres.

$kKZ$ in all these variables

Then: this is quantum difference equation

$qKZ$: Ingredients: unitary $R$ matrix, $R_{21}(u^{-1})R_{12}(u) = 1$.

$+ YB$. 

a collection of operators $Z$ such that $[Z \otimes Z, R] = 0$

e.g. an element in the maximal torus of our quantum group $Z \otimes Z = \Delta Z = \Delta_{opp} Z$

get a representation of the affine Weyl group of type $GL(l)$

acting in $V_1(a_1) \otimes V_2(a_2) \otimes \ldots \otimes V_l(a_l) = \text{functions of } a_1, \ldots, a_l$

with values in $V_1, \ldots, V_l$.

unitarity $\Rightarrow (R^v)^2 = 1$.

$YB \Rightarrow$ Coxeter relation

so we get an action of $S(l)$ to get affine Weyl group

representation $Waff$

quasi periodic spin chain

still a representation of $Waff$

in XXZ spin chain $(\frac{10}{0}) = Z$

$(0,0,1,0,0 \ldots)$.
To produce commuting $q$-difference operators we take $Z_{\text{new}} V_i(a_i) = [Z_{\text{old}} V_i](qa_i)$.

Still, $[Z \otimes Z, R] = 0$ because $R V_1(a_1) V_2(a_2) = R V_{1,2}^{a_1/a_2}$ does nothing here.

for $V_1(a_i) \otimes V_2(a_2)$ in the maximal torus

$$\Psi(q_{a_1}, a_2) = (Z \otimes 1) R_{1,2}^{a_1/a_2} \Psi$$

How can something like this be the quantum difference eq.?

The home of $Z$

Maximal torus of $U_q(\mathfrak{g})$/center $\simeq \text{Pic}(X) \otimes \mathbb{C}^*$

Monomial in $Z$

Only "constant" maps contribute.

Contains the summation over all degrees with $Z$ deg.

Series in $Z$.