Lecture 26, 11/10/2020

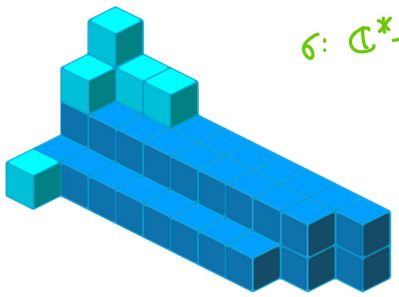
Tuesday, November 10, 2020 10:15 AM

Enumerative geometry & geometric representation theory

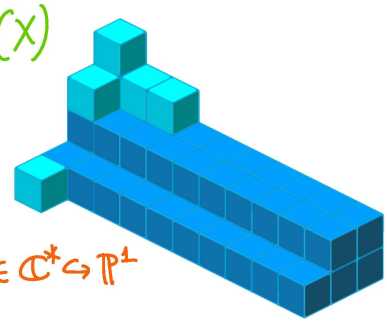
Start time Moscow ~~17:30~~ 18:30

New York 10:30

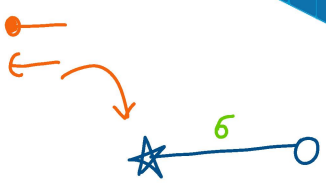
$\sigma: \mathbb{C}^* \to T \subset \mathrm{Aut}(X)$



vs.

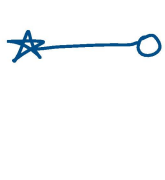


fixed by $q \in \mathbb{C}^* \hookrightarrow \mathbb{P}^1$



$=$

last time



$t \mapsto \sigma(q)t \qquad t \in T$

from the difference in pure edge

$\frac{\Gamma}{\Gamma}$

$z^{\deg}$

degree of the "constant map"

in example, get z^{-4}

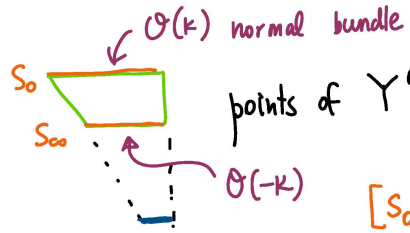
a 1-parameter subgroup

$\sigma: \mathbb{C}^* \to \mathrm{Aut}(Y)$, e.g. $Y = \mathbb{P}^1$.

$\sigma: z \to z^k \hookrightarrow \mathbb{P}^1$

gives

$$\begin{array}{ccc} Y & \hookrightarrow & \tilde{Y} \\ & & \downarrow \\ & & \mathbb{P}^1 \end{array}$$



points of Y^σ give "constant sections"

$[S_0] - [S_\infty] = k\,[\text{Fiber } \mathbb{P}^1]$

Measure the degrees of curves by pairing them with line bundles

$[\text{curves}] \subset \mathrm{Pic}^\vee$

degree in $\mathbb{P}^1$ direction $= 1$.

$$0 \to \underset{\substack{\| \\ \mathrm{Pic}(\mathbb{P}^1)}}{\mathbb{Z}} \to \mathrm{Pic}(\tilde{Y}) \to \mathrm{Pic}(Y) \to 0.$$

$$\langle \mathcal{L}, \text{constant section} = y \in Y^{\sigma} \rangle = \langle \sigma, \mathcal{L}|_y \rangle$$

$\mathbb{P}^1$: $0 \xrightarrow{q} \infty$

σ: cocharacter; $\mathcal{L}|_y$: character

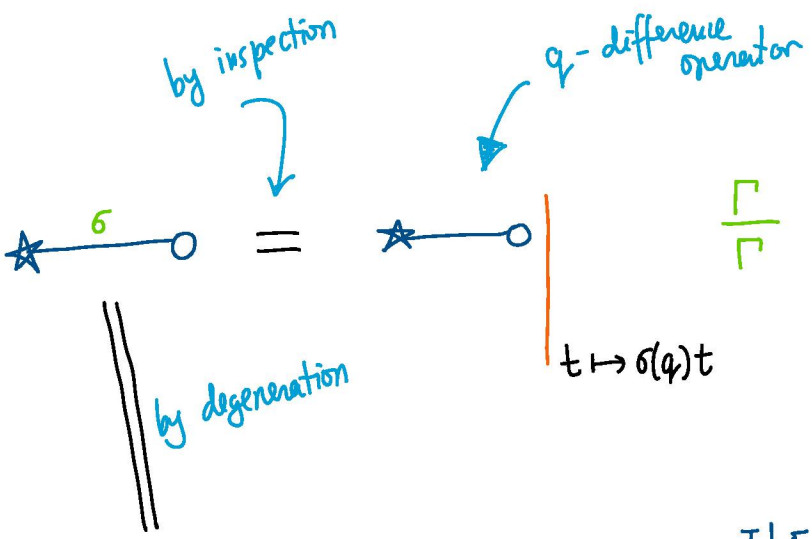
$\langle \mathcal{L}, \text{constant section} \rangle = \deg_{\mathcal{L}}(\text{constant section})$

in example $\mathcal{O}(1)|_0 = q \cdot q^{\ell}$ $\qquad$ $\mathcal{O}(1)|_\infty = 1 \cdot q^{\ell}$

$\sigma(z) = z^k = q$ $\qquad$ $\langle \sigma, \cdot \rangle$ these differ by k

$$-4 = \langle (t_1, t_2), \mathcal{O}(1)|_{\boxplus} = t_1^{-4} t_2^{-2} \rangle$$

$(t_1, t_2) = (q, 1)$



$$\frac{\Gamma}{\Gamma} \quad z^{\langle \sigma, - \rangle}$$

degree of the constant map

This means: Ψ = the operator $\longleftarrow\!\circ$

$= \Psi(z, t, q)$

$$\underbrace{\overset{\sigma}{\longleftrightarrow} \text{ Glue}^{-1}}_{S_\sigma(z,t,q)} \quad \Psi(z,t,q) = \Psi(z, \sigma(q)t, q) \frac{\Gamma}{\Gamma} z^{\langle \sigma, - \rangle}$$

this is a q-difference eq. in equivariant variables

$S_\sigma(z,t,q)$

rational function
(hard to prove)

REAL WORK TO IDENTIFY

$\Gamma_q(w^{-1})\Gamma_q(\hbar w) \approx \theta(w)$

this is a q-difference eq. in equivariant variables

Better: Ψ intertwines an interesting q-difference connection defined by S_σ .... with the abelian connection corresponding to

very closly connected with "attractive" line bundles $\mathcal{J} = \Theta(T^{1/2}_X)\otimes\mathcal{U}$ in the discussion of elliptic stable envelopes

produces z_i^{-1} under $\mathcal{L}_i \to q\mathcal{L}_i$

$$\prod \frac{\theta(z_i\mathcal{L}_i)}{\theta(z_i)\,\theta(\mathcal{L}_i)} = \Theta\left(\sum (z_i-1)(\mathcal{L}_i-1)\right)$$

the corresponding coordinates on $\mathrm{Pic}\otimes\mathbb{C}^*$

$\mathcal{L}$ a basis of Pic

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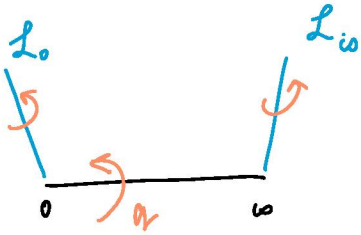
q-difference equation in Kähler variables



by localization or by the structure of $K(\mathbb{P}^1) = \cancel{\ }$ we have:

$$c_1(\mathcal{L}_0) - c_1(\mathcal{L}_\infty) = \deg\mathcal{L}\cdot c_1(T_0\mathbb{P}^1).$$

the weight

q

$$\mathcal{L}_0\otimes\mathcal{L}_\infty^{-1} = (T_0\mathbb{P}^1)^{\deg\mathcal{L}}.$$

F is any sheaf on $\mathbb{P}^1$, $\det F$ is a line bundle, $\deg F = \deg\det F$

$$q^{\deg F} = \frac{\det F_0}{\det F_\infty}$$

the same formula for the degree of the constant map

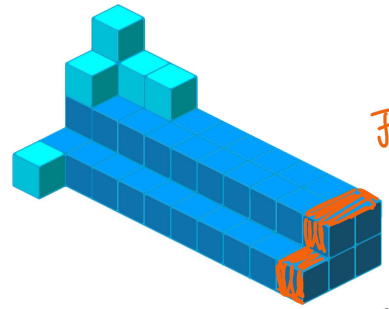
$$\chi\Big(\mathcal{M},\ z^{\deg}\,\hat{\mathcal{O}}_{vir}\otimes \text{insertions}\Big)\Big|_{z\to qz} =$$

moduli of maps

$$= \chi\Big(\mathcal{M},\ z^{\deg}\,\hat{\mathcal{O}}_{vir}\otimes \text{insertions}\otimes \det(\mathcal{F}|_0)\otimes\det(\mathcal{F}|_\infty)^{-1}\Big)$$

moduli of maps

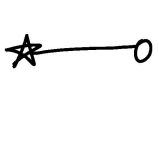
$\mathcal{F}_\infty = \mathcal{F}\otimes\mathcal{O}_\infty$.



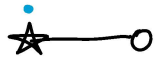
pulled back by evaluation map.

if $\mathcal{M} = \{$ ★ ⟵ ○ $\}$ then

0 ∞



$\Big|_{z\mapsto qz}$ $=$

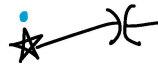


$\otimes\ \mathcal{L}^{-1}$

line bundle on X corresponding to z in this case $\mathcal{O}(1)$.

where $\bullet = \det(\mathcal{F}\otimes\mathcal{O}_0)$.

by degeneration



$\otimes\ \mathcal{L}^{-1}$

Recall $\Psi =$ ⟵ ○

$$\boxed{\Psi(q^{\mathcal{L}}z,t,q)\otimes\mathcal{L} = M_{\mathcal{L}}\,\Psi(z,t,q)}$$

$M_{\mathcal{L}} =$ ⟷ $\cdot\ \mathrm{Glue}^{-1}$.

$z\in \mathrm{Pic}(X)\otimes\mathbb{C}^*$.

$\mathrm{Pic} =$ cocharacters

q- difference equation in Kähler variables

Main feature: commutes with the

... in equiv. variables

Main feature: commutes with the q-diff. conn. in equiv. variables

a solution to a q-diff. eq.

in the current logic, this is how it is determined

Goal: describe this flat q-difference connection in quantum group terms

Plan: Start with equations for certain special equivariant variables

Motivation:

$$K(T^*Gr(k,n)) = \text{integral form of weight } k \text{ subspace in}$$

$$\mathbb{C}^2(a_1)\otimes\mathbb{C}^2(a_2)\otimes\dots\otimes\mathbb{C}^2(a_n)$$

as $U_\hbar(\widehat{\mathfrak{gl}}(2))$ module where

evaluation representation.

$$\begin{array}{c} k \quad V \\ \updownarrow \\ n \quad W \end{array} \qquad \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \in GL(n) = GL(W)$$

changes of framing

qKZ connection moves a_i around.

Clearly this must be the connection we want, no?

More generally:

$$Q = \begin{array}{ccccc} V_1 & \rightleftarrows & V_2 & & \dots \\ \updownarrow & & \updownarrow & & \\ W_1 & & W_2 & & \dots \end{array}$$

$$GL(W) = \prod GL(W_i).$$

$$K(\mathcal{M}_Q(v,w)) = \text{integral form of the weight } v \text{ subspace in } \bigotimes_i \bigotimes_{j=1}^{w_j} \text{ ith fund eval. repres. } (a_{ij})$$

qKZ in all these variables

Thm: this is quantum difference equation

qKZ: Ingredients: a collection of: unitary R matrix, $R_{21}(u^{-1})R_{12}(u) = 1$.

+ YB

YBE

a collection of.

+ YB

operator Z such that $[Z\otimes Z, R] = 0$

e.g. an element in the maximal torus of our quantum group

$Z\otimes Z = \Delta Z = \Delta_{opp} Z$

$\Downarrow$

get a representation of the affine Weyl group of type $GL(\ell)$

acting in $V_1(a_1)\otimes V_2(a_2)\otimes \dots \otimes V_\ell(a_\ell) =$ functions of $a_1,\dots,a_\ell$ with values in $V_1,\dots,V_\ell$.

$\check{R} = (12)\circ R(a_1/a_2)$.

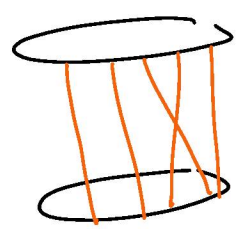
$V_2(a_2)\otimes V_1(a_1)\otimes \dots$

unitarity $\Rightarrow (\check{R})^2 = 1$.

YB $\Rightarrow$ Coxeter relation

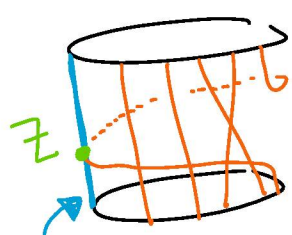
so we get an action of $S(\ell)$

to get affine Weyl group



periodic spin chain

representation W_{aff}



quasi periodic spin chain

seam of the cylinder

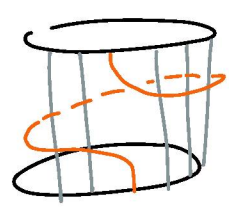
add rotation by 1.

still a representation of W_{aff}

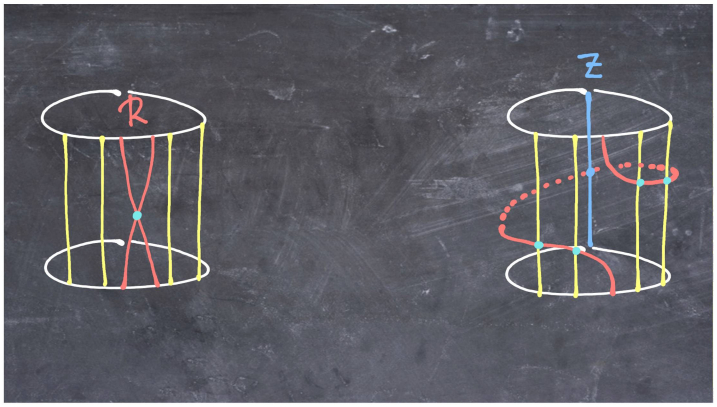
in XXZ spin chain $\begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} = Z$.

$\tilde{W}_{aff} = S(\ell) \ltimes \mathbb{Z}^\ell$

commuting operators



$(0,0,1,0,0\dots)$.



To produce commuting q-difference operators we take $Z_{new} V_i(a_i) =$

$$= [Z_{old} V_i](q a_i)$$

still, $[Z \otimes Z, R] = 0$ because $R_{V_1(a_1), V_2(a_2)} = R_{V_1, V_2}(a_1/a_2)$

$a_1 \to q a_1$, $a_2 \to q a_2$

does nothing here.

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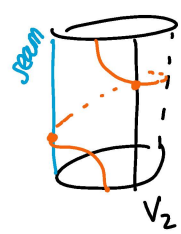
for $V_1(a_1) \otimes \underbrace{\ldots\ldots}_{\text{new } V_2(a_2)}$



in the maximal torus

$$\Psi(q a_1, a_2) = (Z \otimes 1) R_{12}(a_1/a_2) \Psi$$

$(a_1, a_2, \ldots) = \sigma = (q, 1, \ldots)$

$\longleftrightarrow$ Glue^{-1}

¿How can something like this be the quantum difference eq?

contains the summation over all degrees with $z^{\deg}$

the home of z

monomial in z $\longleftrightarrow$ Series in z.

only "constant" maps contribute.

maximal torus of $U_q(\mathfrak{g})$ / center $\simeq \mathrm{Pic}(X) \otimes \mathbb{C}^*$.