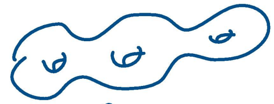
Lecture 24, 10/27

Tuesday, October 27, 2020 10:30 AM

Enumerative geometry & geometric representation theory

Start time Moscow 17:30

New York 10:30

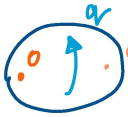
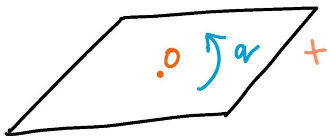
Basic pieces of enumerative theory of  $C \to X \subset \mathfrak{X}$ quotient stack

$\mathrm{Hilb}(\mathbb{C}^2, n)$

Nakajima variety

symplectic reduction

....

$C =$  $=$  + boundary condition at ∞

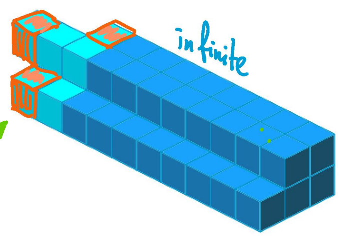
$\mathbb{P}^1$ $\quad$ $\mathbb{C}$

$\bullet\!\!-$ evaluation to $\mathfrak{X}$

$\circ\!\!-$ maps nonsingular at this point, evaluation to X

$\leftarrow$ relative point, accordions open, proper evaluation to X

$\bullet\!\!-\!\!\circ$



a point $\mathrm{Hilb}(\mathbb{C}^2, 5)$.

fiber at ∞ $\in$ stack

"1-legged vertex" with descendants (at 0)

characteristic classes of $\mathcal{F} \otimes \mathcal{O}_0$

$\mathrm{Hilb}(A_n)$ for more legs

bubbles

$\leftarrow\!\!-\!\!\circ$

By degeneration: $\bullet\!\!-\!\!\circ \;=\; \bullet\!\!-\!\!\times\!\!-\!\!\circ$

the matrix that connects $\bullet\!\!-\!\!\circ$ and $\leftarrow\!\!-\!\!\circ$

$\bullet\!\!\to$

$\infty \times p(n)$ matrix

$p(n) \times p(n)$ matrix

rank of $K(\mathrm{Hilb}(\mathbb{C}^2, n))$

By localization: $\longleftrightarrow \;=\; \longleftarrow\!\circ \;\; \circ\!\longrightarrow$

in fact $= \longleftarrow m \longrightarrow$

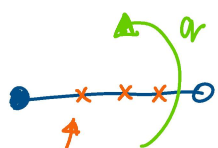
So, a bunch of interrelated tensors in $K(X)^{\otimes 2}$ or $K(X) \otimes K(\mathfrak{X})$, ...

$\bullet\!\!-\!\!\circ$ view as $ev_{0,*}\left(ev_\infty^*(\cdot)\cdot \hat{\mathcal{O}}_{vir}\, z^{\deg}\right) \in K(\mathfrak{X})_{loc}\,[[z]]$

$K(X)$

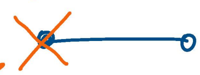
convergent

not proper
localization w.r.t
$\mathbb{C}_q^\times \curvearrowright \mathbb{P}^1$
is required



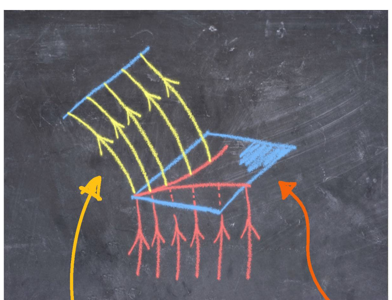
base pts

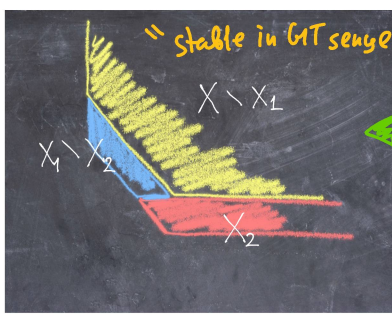
fixed loci



big fat base pt

Claim: This map, with suitable q-Gamma function factors is the elliptic stable envelope in the context of $X \subset \mathfrak{X}$





" stable in GIT sense, not attracted " $= X \setminus X_1$

stratification of X by
Attr (components of X^A)

long exact sequence
in elliptic cohomology

a class on the open
in this instance
Attr $\subset$ component of $X^A \times \left(X \setminus \text{lower attracting manifolds}\right)$

Now: GIT stable locus
$X \subset \mathfrak{X}$

$\mathfrak{X} \setminus X$ = has a Bogomolov–Hesselink–Kempf–Ness–Rousseau–… stratification

quotient by G, e.g. $G = \prod GL(V_i)$ may be reduced to tori by a trick

reductive connected (will explain)

$\mathfrak{X} = \left[\text{prequotient} / G\right]$

Stab: $\Theta\left(T^{1/2}\,\text{prequotient}\big|_X\right) \otimes$ degree 0 depends on z $\longrightarrow \Theta\left(T^{1/2}\,\text{prequo}\right) \otimes$ —"—

degree 0 depends on z $\in \mathrm{Pic}(X) \otimes \mathbb{C}^*$

restriction

$\mathrm{Ell}_{\mathrm{equiv}}(X)$

for $E = \mathbb{C}^*/q^{\mathbb{Z}}$ — our q

sections of line bundle on $\mathrm{Ell}_{G \times G_{\mathrm{aut}}}(pt)$

closed here, 0-dimensional over $\mathrm{Ell}_G(pt)$

means: interpolate an elliptic function from finitely many values

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$\longleftarrow\!\!\!\circ$ is a fundamental solution of a flat q-difference connection in both Kähler variables $z \in \mathrm{Pic}(X) \otimes \mathbb{C}^*$ and equivariant variables in $\mathrm{Aut}(X)$

same q as before, free parameter!

a generalization of the dynamical equations

this is for $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$, $\hbar$ is not related to q $\longrightarrow$ a generalization of qKZ to a cocharacter $\sigma: \mathbb{C}^* \to \mathrm{Aut}(X)$

$a \mapsto \sigma(q)\,a$

≈ "R-matrix"

Very classical question: can one solve this equation by an $\int$?

in usual hypergeometric world "Euler" $= \int_0^1 x^{\cdots}(1-x)^{\cdots}(1-xz)^{\cdots}\,dx$

in q-difference situation "Mellin-Barnes" $= \int \mathrm{poly}(x)\,\frac{\Gamma(\ldots)}{\Gamma(\ldots)}\,x^z\,dx$

"Mellin-Barnes" $= \int \mathrm{poly}(x) \dfrac{\Gamma(\dots)}{\Gamma(\dots)} x^{\cdot} \, dx$

in q-difference situation

$$(1-x)^m \rightsquigarrow (1-x)(1-qx)\dots(1-q^{m-1}x) = \frac{(x)_\infty}{(q^m x)_\infty} \qquad (x)_\infty = \frac{1}{\Gamma_q(x)} = \prod_{i=0}^{\infty}(1-q^i x)$$

$|q|<1$

$$\Gamma_q(x) = \frac{1}{\prod_{i\geq 0}(1-q^i x)} = \text{character of } \mathbb{C}[\text{ generators of weight } x, qx, q^2x, \dots]$$

$\| \; \mathcal{O}_{\mathrm{Maps}(\mathbb{C}\to\mathbb{C})}$ with $q \circlearrowleft$, $x^{-1} \circlearrowleft$

dual

coordinate t on $\mathbb{C}$

$t^k \longrightarrow x^{-1}q^{-k}t^k$

really ubiquit...

$$\left(\alpha \, , \begin{array}{c}\text{fundamental} \\ \text{solution of} \\ \text{q-difference eq}\end{array} \beta \right) = \text{usually} \int_{\text{cycle}(\beta)} f_\alpha(x,\dots) \,\text{weight}(x,\dots)\, dx$$

dummy integration

real variables in my equation

vectors in a repr. of our quantum group, concretely $K_{eq.}(X)$

elliptic function

$$= \text{better} \int_{|x_i|=1} f_\alpha(x,\dots)\, g_\beta(x,\dots) \,\text{weight} \prod \frac{dx_i}{2\pi i x_i}$$

max torus in G

includes Weyl $\int$ formula + q-Gamma functions

in our situation, this takes the form:

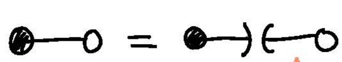
K-theoretic stable envelope(α)

Elliptic stable envelope (β)

from Maps($\mathbb{C} \to$ Prequotient)

inner product in $K(\mathcal{X})$

$(\mathcal{F}_1, \mathcal{F}_2) \to \chi(\text{prequotient}, \mathcal{F}_1 \otimes \mathcal{F}_2)^G$



This is my fund solution.

If I can find f_α such that

solved by stable envelopes in K-theory, as we will see

then

Then $\quad \alpha \longleftarrow\!\circ \;=\; \overset{\cdot\alpha}{\bullet}\!\!-\!\!\!-\!\!\circ$ $\qquad$ in ... we will see

$$\alpha \longleftarrow\!\circ\,\beta \;=\; \overset{f_\alpha}{\bullet}\!\!-\!\!\!-\!\!\circ\,\beta \;=\chi\left(QM, ev_0^*(f_\alpha)\, ev_\infty^*(\beta)\, \hat{\mathcal{O}}_{vir}\, z^{\deg}\right).$$

$$=\chi\left(\mathcal{X},\, f_\alpha \otimes \text{elliptic stable}(\beta) \otimes ev_{0,*}\left(\text{maps } \mathbb{D} \to X\right)\right).$$

the origin of Γ_q factors in the formula $\bullet\!\!-\!\!\!-\!\!\circ$.

$$=\int f_\alpha(x)\, g_\beta(x)\ \Gamma_q\text{-functions } dx.$$

Suppose

$$\int_{|x_i|=1} f_\alpha(x, z)\, g_\beta(x, z)\ \text{weight} \prod \frac{dx_i}{2\pi i x_i}$$

critical point

$\nabla_x \ldots = 0$

Bethe equations.

"off shell Bethe eigenvector"

the eigenvector

solves an equation of the form $\Psi(qz) = M(z)\Psi(z)$

what will happen if $q \to 1$

eigenvector of $M(z)$ with eigenvalue $\lambda_i(z)$

$$\Psi(z) \sim e^{\frac{1}{\ln q}\int \lambda_i}\, \psi_i(z)$$

this limit tells me both eigenvalues and eingenvectors of $M(z)\big|_{q=1}$

$f_\alpha(x) = $ insertion at 0

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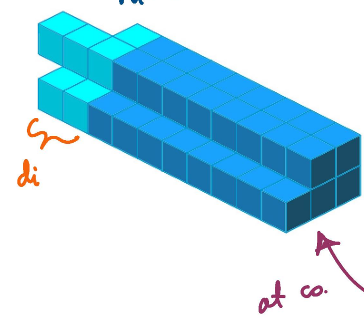

$$\int_{|x_i|=1} f_\alpha(x,\dots)\, g_\beta(x,\dots) \text{ weight} \prod \frac{dx_i}{2\pi i x_i}$$

singularities are here

$$g_\beta(q^{d_i}x) = z^{\sum d_i}\, g_\beta(x).$$

$z^{\deg.}$

for Hilb$(\mathbb{C}^2,n)$ they

$$\frac{\cdots\cdots}{\underbrace{\varphi(1/x_i)\,\varphi(t_1 x_i/x_j)}\ \underbrace{\varphi(q x_i/t_1 x_j)}}$$

Poles of the form

$$x_i = q^{d_i}\, t_1^{\cdots}\, t_2^{\cdots}$$

Chern roots of $\mathcal{F}\otimes\mathcal{O}_0$.

poles we don't like

killed by g_β

because g_β is supported on $\mu^{-1}(0)\subset\mathcal{X}$

$$g_\beta\big|_{\mathcal{X}\setminus\mu^{-1}(0)} = 0 \qquad \mathrm{Ell}(\mathcal{X}\setminus\mu^{-1}(0)) \xhookrightarrow{\text{closed}} \mathrm{Ell}(\mathcal{X}).$$

" wheel conditions "

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elliptic curve of K-theory

$$\mathbb{C}^\times \longrightarrow \mathbb{C}^\times/q^{\mathbb{Z}} = E$$

elliptic curve of elliptic cohom theory

$$\forall X,G \qquad \operatorname{Spec} K_G(X) \longrightarrow \mathrm{Ell}_G(X)$$

" reduction mod q "

Solution of some abelian q-difference eq. $\longleftarrow$ Section of line bundle

$\vartheta(x)$ as a function of $x\in\mathbb{C}^*$ satisfies

something like $\vartheta(qx) = -q^{-1/2}x^{-1}\vartheta(x)$

$\vartheta(x)$ as a function of $x \in \mathbb{C}^{\times}$ satisfies

something like $\vartheta(qx) = -q^{-1/2} x^{-1} \vartheta(x)$