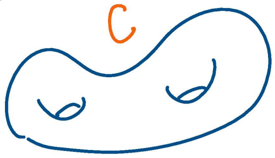
Lecture 22, 10/13

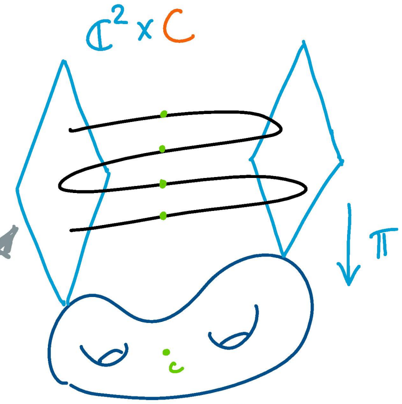
Tuesday, October 13, 2020 9:37 AM

Enumerative geometry & geometric representation theory

Start time Moscow 17:30
New York 10:30

$$\left\{ \text{C} \quad \xrightarrow[\text{map}]{f} \mathrm{Hilb}(\mathbb{C}^2, n) \right\} = \left\{ \quad \right\}$$





no singularities

parametrizes 0-dimensional subschemes of $\mathbb{C}^2$ of length n

Subscheme flat over C

$\mathbb{C}[x_1, x_2]1 = \mathcal{O}_{\mathbb{C}^2} \to V = \mathcal{O}_{\text{subscheme}}$

fiber over c = lim nearby

$C = \mathbb{P}^1$

has to converge to a map that is constant outside of $0 \in \mathbb{P}^1$ " $f(\infty)$

Quasimaps to $\mathrm{Hilb}(\mathbb{C}^2, n)$ relax the surjectivity

$$\mathcal{O}_{C \times \mathbb{C}^2} \xrightarrow{s} \mathcal{O}_{\cancel{\text{subscheme}}} \to 0 \qquad \mathcal{F}$$

V = vector bundle on C $= \pi_* \mathcal{F}$ $\circlearrowleft$ x_1, x_2 $\iff$ no 0-dimensional subsheaves in $\mathcal{F}$

$x_1^i x_2^j \cdot 1$ generate V except at finitely many base pts $\iff$ $\dim \mathrm{Coker}(s) = 0$

Pandharipande–Thomas: $\underbrace{\mathcal{O}_{3\text{-fold}} \xrightarrow{s} \mathcal{F}}$

- $\mathcal{F}$ pure dimension $= 1$
- $\mathrm{Coker}(s)$ $\dim = 0$

$\cap$

" Donaldson-Thomas counts "

count stable objects in categories that look like sheaves on 3-folds

$\mathrm{Ext}^i(\mathcal{F}_1,\mathcal{F}_2) = \mathrm{Ext}^{3-i}(\mathcal{F}_2,\mathcal{F}_1\otimes K)^\vee$

$\mathrm{Ext}^1(\mathcal{G},\mathcal{G})$ = deformations

$\mathrm{Ext}^2(\mathcal{G},\mathcal{G})$ = obstructions

$\approx$

- sheaves on 3-folds, in particular curves
- representations of quiver in sheaves over a curve $\approx$ quasimaps

[mnop] : ideal sheaves of curves in 3-folds

$$\mathcal{O}_{3\text{-fold}} \to \mathcal{O}_{\text{curve}} \to 0$$

means giving up on $\mathcal{V}$ being a vector bundle

tons of other possibilities for stability conditions

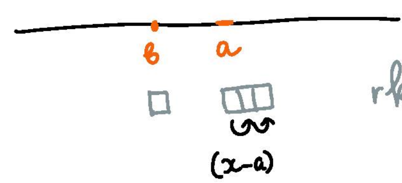
Expectation (supported by many theorems / computations)

- all stability conditions give equivalent counts (known for local curves in cohomology, in K-theory area of active current research)
- in cohomology, equivalent to GW counts of curves (known for local curves).

(in K-theory ???) $\leftarrow$ the loss of self-duality seems like a problem to me

what PT/DT looks like for $\mathbb{C}$ instead of $\mathbb{C}^2$

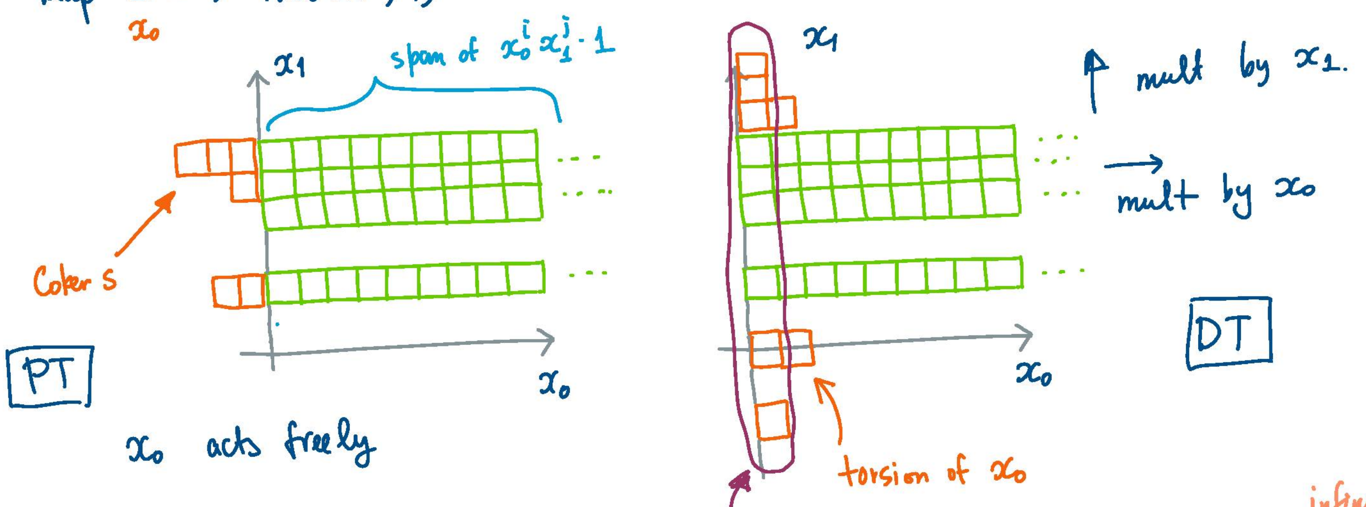
Subscheme of $\mathbb{C}$ e.g. $(x-a)^3(x-b)$



draw Jordan blocks of x

$\mathrm{rk}\,\mathcal{O}_{(x-a)^3(x-b)} = 4$

map $\underset{x_0}{\mathbb{C}} \to \mathrm{Hilb}(\overset{x_1}{\mathbb{C}^1}, 4)$ constant // outside of $0 \in \mathbb{P}^1$



x_0 acts freely

generates everything under the action of x_0

infinite, in this

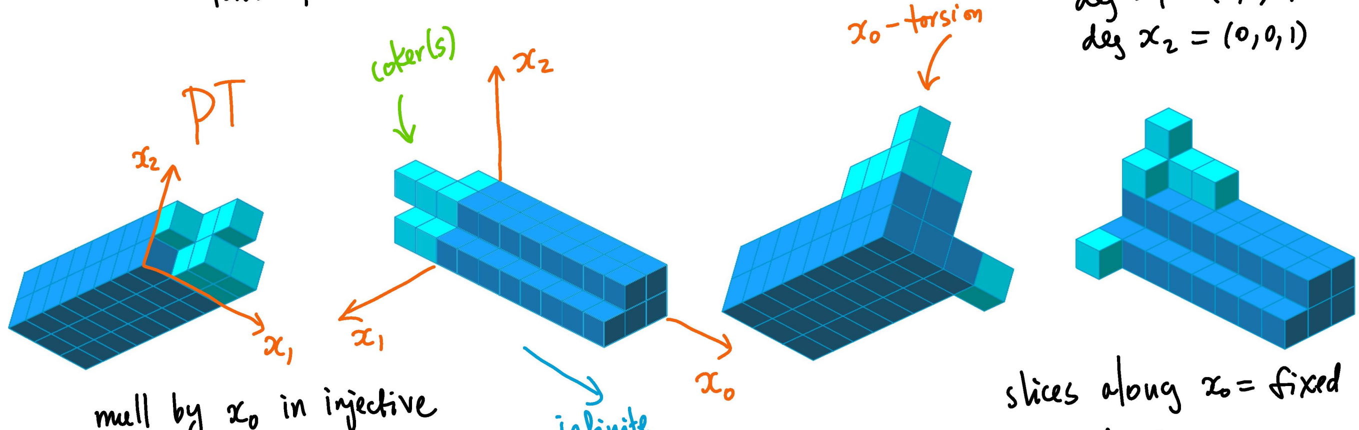
torsion of x_0

generates everything under the action of x_0

infinite in this direction

now in 3D, we will draw torus fixed points $= \mathcal{F}$ is graded by $\mathbb{Z}^3$ and

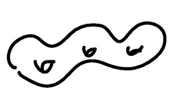
$\deg x_0 = (1,0,0)$
$\deg x_1 = (0,1,0)$
$\deg x_2 = (0,0,1)$



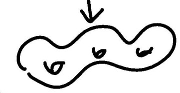
mult by x_0 in injective slices <u>increase</u>

slices along x_0 = fixed <u>decrease</u>

gets a lot more interesting for $\mathbb{C}^2 \rightsquigarrow$ ADE surface, see H. Liu "Quasimaps and stable pairs"

3-fold  $\times\ \mathbb{C}^2 \rightsquigarrow$ $\mathcal{L}_1 \oplus \mathcal{L}_2$ (coord.) $x_1 \in \mathrm{Hom}(V,V) \otimes \mathcal{L}_1^\vee$

$\downarrow$



General principle $X \subset \mathcal{X} = \left[\text{prequotient} / G \right]$

$1 \to G \to \widetilde{G} \to G_{Aut} \to 1$

$\cup$

$GL(2)$ for $\mathrm{Hilb}(\mathbb{C}^2, n)$

instead of $\mathcal{P}$ = principal G-bundle

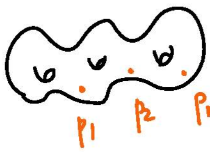
can consider $\widetilde{\mathcal{P}}$ = principal $\widetilde{G}$-bundle with given image in G_{Aut}-bundles.

$\approx$ quasimaps

"Counts of curves in X" $\Longrightarrow$ tensors in $K_{eq}(X)^{\otimes \cdots} \otimes K_{eq}(\mathcal{X})^{\otimes \cdots}$

to compute it means to identify in terms of quantum groups by G_{aut}

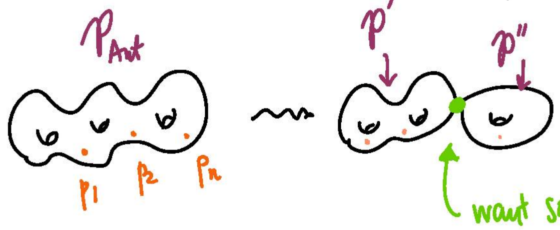
twisted stable quasimaps to $X \subset \mathfrak{X}$

$QM \Big\{$  $\xrightarrow{f} X \Big\} \xrightarrow{ev} (f(p_1), f(p_2), \dots) \in \mathfrak{X}^n$

$p_1 \quad p_2 \quad p_n$

"extended" K-theoretic CohFT

$\Big(\mathcal{P}_{Aut}$  $, \; f(p_1), \dots, f(p_n) \Big) \in \mathcal{B}un \times \mathfrak{X}^n$



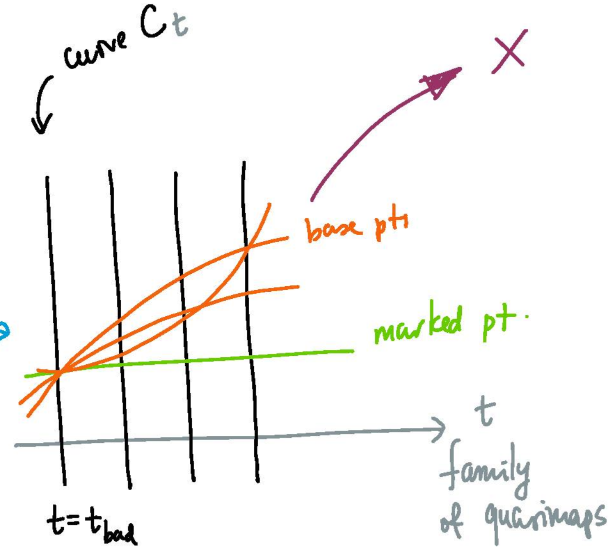
want some gluing here, nodes have to be disjoint from base pts

to get an evaluation map to X, not to $\mathfrak{X}$, we need a moduli space that resolves the following map.

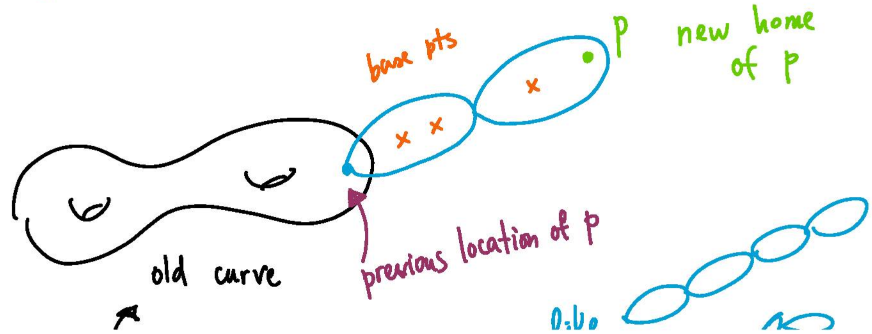
$$QM \overset{\text{open}}{\supset} QM_{\substack{\text{nonsing}\\ \text{at } p}} = \{ f(p) \in X \}$$

"blow up"

$QM_{\substack{\text{relative}\\ \text{at } p}}$ $\xrightarrow{\text{regular proper map}}$ $f(p) \in X$



blow up
(weighted blow up, base changes, ...)

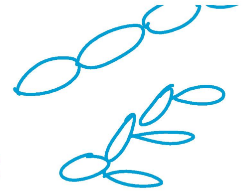


New curve has " bubbles

old curve

↑ fixed, parametrized

previous location of



only opens like

not like

New curve has $\mathrm{Aut} \simeq (\mathbb{C}^\times)^{\#\text{ bubles}}$

mod out by it

new stability $= \left\{ \begin{array}{l} \dim(\text{base locus}) = 0 \\ \text{and} \quad \dim(\text{Stabilizer}) = 0 \end{array} \right\}$ prevents unnessary bubbles

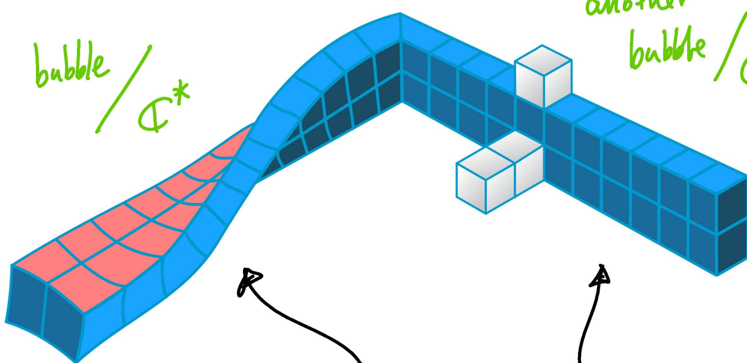
Note: Relative moduli space is a DM stack

localization is complicated fixed loci are not isolated

Example complicated/nonisolated fixed loci for relative quasimaps to $\mathrm{Hilb}(\mathbb{C}^2, 2)$

bubble / $\mathbb{C}^*$

another bubble / $\mathbb{C}^*$

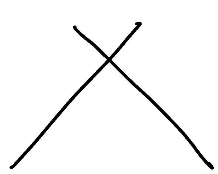


(1) The action of $(\mathbb{C}^*)^2$ may be compensated by rescaling $(\mathbb{C}^*)^{\#\text{ of bubbles}}$ in the domain, like in this bubble. This is called "twister"

(2) Since in the bubble we loose the action on the domain, one gets a lot of fixed loci of the form called "skewer"

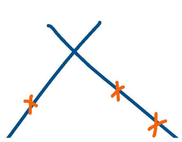
In both cases, we can have finite stabilizers in $(\mathbb{C}^\times)^{\#\text{ of bubbles}}$, so really orbifold points in QM relative

$$\mathrm{QM}\left(\vphantom{\int} \right) = ?$$

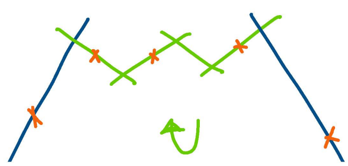


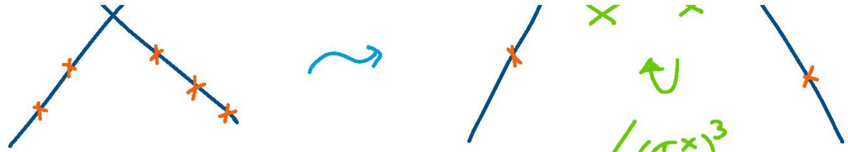
nodes $\cap$ basepoints $= \emptyset$

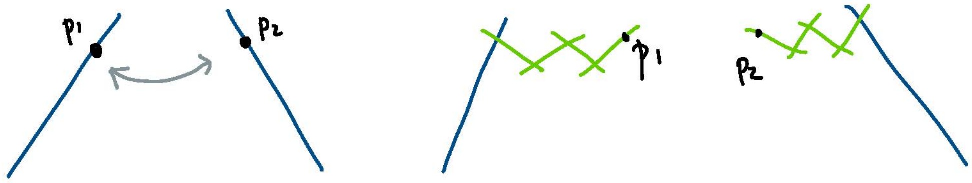
accordion swallows base pts



$\rightsquigarrow$





vs 

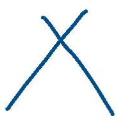
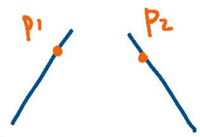
just "glue" means pair with diagonal $\in X^2 \ni (p_1, p_2)$ $\rightsquigarrow$ gives overcounting

$K(X)^{\otimes 2}$

$\cup\!\!\!\mid$

diag + corrections

$\parallel$

 $= \Big\langle$  $, \ \mathrm{Glue}^{-1} \Big\rangle$ Glue = 

$\cap\!\!\!\mid$ $K(X)^{\otimes 2} \otimes \ldots$ $\big(K(X)^{dual}\big)^{\otimes 2}$ $/(\mathbb{C}^\times)^5$