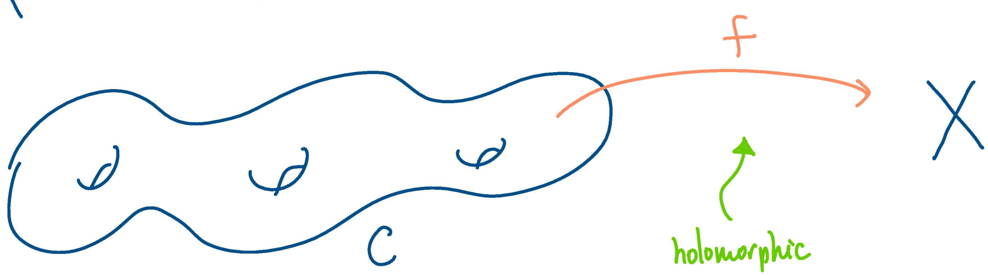
Lecture 21, 10/6

Monday, October 5, 2020 7:52 PM

Enumerative geometry & geometric representation theory

Start time Moscow 17:30
New York 10:30

$X \approx$ moduli space of vacua in a susy (2+1)-dimensional theory



susy states: $\nearrow$, approximately

index sheaf, should be like the index $\not{D}$ = Dirac operator, e.g. for a Kähler M

$$\not{D} = \bar{\partial} \circlearrowleft K_M^{1/2} \otimes \bigoplus_i \Omega^{0,i} M$$

even/odd

$\hookrightarrow$ $\mathcal{O}_{vir} \otimes K_{vir}^{1/2}$

obstruction theory T_{vir} = Deformation − Obstructions

TM

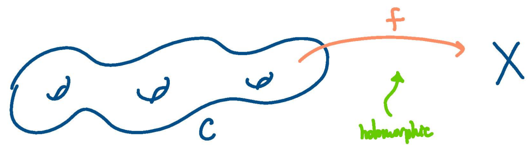
$\det(\ )^{-1} = K_{vir}$

$$\mathcal{O}_{vir} = \mathcal{O}_{Deformation} \otimes \left(\sum (-1)^i \Lambda^i \, Obstr^* \right)$$

Want: Obstructions $\approx$ (Deformations)* $\rightsquigarrow$ $K_{vir} = (\det Obs)^2$

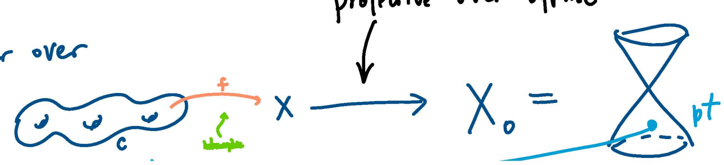
also OK: Obstructions $\approx$ (Deformations)* $\otimes$ something we can control

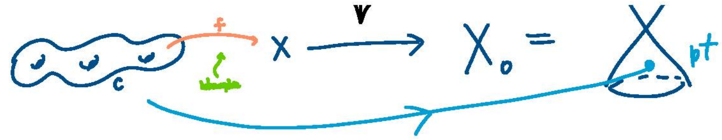
Want a moduli space of maps



- with a perfect obstruction theory that is almost self-dual
- proper over

projective over affine

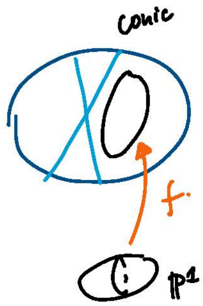




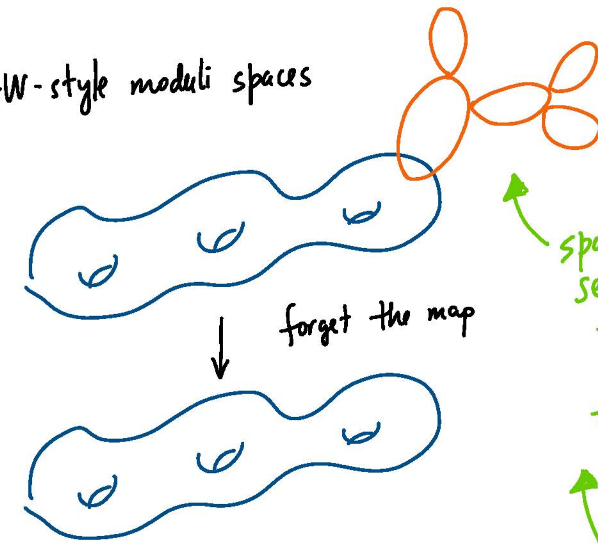
not so easy to achieve, actually

to have a proper moduli space, need to allow maps with singularities

$T^*\mathbb{P}^2 \supset$



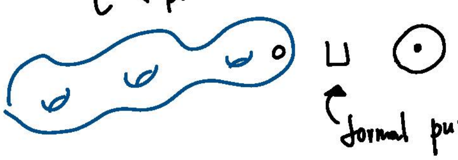
in GW-style moduli spaces

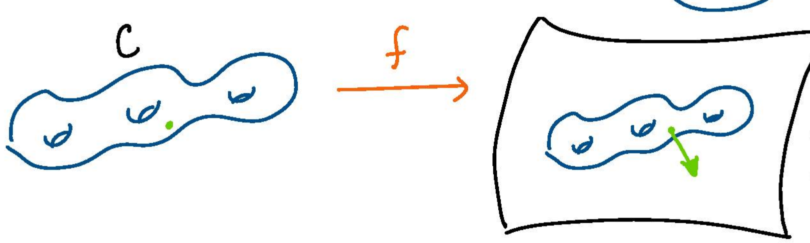


spoil self-duality for the best of targets X

affect K_C in basically unmanageble way

$C \setminus pt$





my personal opinion, not a math statement $\Longrightarrow$

good paper to read by Henry Liu

$$\mathrm{Def} = H^0(C, f^*TX) \qquad \mathrm{Obs} = H^1(C, f^*\underline{\underline{TX}})$$

self dual, always, for X of the kind we consider

$$\mathrm{Obs}^\vee = H^1(C, f^*T)^\vee \overset{\text{Serre}}{=}$$

$$= H^0(C, \cancel{\text{dual}}(f^*T) \otimes K_C)$$

can control as long as we can control C

Basic challenge in K-theoretic GW theory

$$X = T^*G/B$$

$\hookrightarrow$ Macdonald operators

$\hbar \to 0, \infty$

$$X = G/B$$

Toda (Givental and Y.P. Lee)

Not going that way in this course.

$X \leftsquigarrow$ susy gauge theory   e.g. Nakajima quiver variety

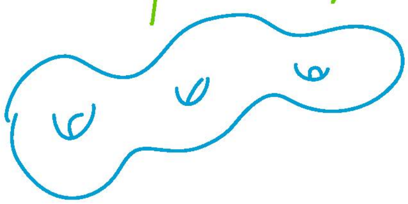
(note: T^*G/B is not a Nakajima variety for $G \neq GL(n)$).

$$\underset{\text{stable locus}}{X \subset} \mathfrak{X} = \left[\mu^{-1}(0) / G \right]$$

where $G \hookrightarrow$ symplectic repr M

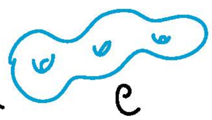
gauge group   matter

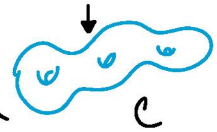
$$\mu : M \to \mathfrak{g}^\vee$$



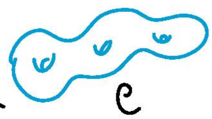
$\mathcal{P}$ = a principal G-bundle over C

$$\left\{ C \to \mathrm{pt}/G \right\} = \left\{ \mathcal{P} \downarrow C \right\}$$





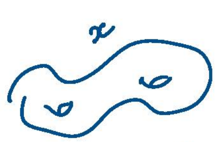
$$\left\{ C \to Y/G \right\} = \left\{ \mathcal{P} + \text{section of } \mathcal{P} \times_G Y \downarrow C \right\}$$



for instance $G = \mathbb{C}^\times, GL(1), \mathbb{G}_m \hookrightarrow Y = \mathbb{C}^n, \mathbb{A}^n, \dots$

$$\left\{ \text{line bundle } \mathcal{L} \downarrow C + \text{section of } \mathcal{L}^{\oplus n} \right\}$$

i.e. n sections of $\mathcal{L}$



$$x \in C \xrightarrow{f} \mathbb{P}^{n-1}$$

sections of $\mathcal{O}(1)$ on $\mathbb{P}^{n-1}$

$$f(x) = [f_1(x) : f_2(x) : \dots : f_n(x)]$$

$\mathcal{L} = f^* \mathcal{O}(1)$ + n sections of $\mathcal{L}$ + without base points

$f(x) = [0 : \dots : 0]$

$$\mathbb{P}^{n-1} \underset{\text{stable}}{\subset} [\mathbb{C}^n/\mathbb{C}^*]$$

$$\mathrm{Maps}\left(C \to [\mathbb{C}^n/\mathbb{C}^*]\right) = \text{a line bundle } \mathcal{L} + n \text{ sections}$$

$$\cup$$

stable Quasimaps $\left(C \to [\mathbb{C}^n/\mathbb{C}^*]\right) = \quad — // — \quad +$ base locus 0-dimensional and disjoint from nodes of C

$$\cup$$

$\mathrm{Maps}\left(C \to \mathbb{P}^{n-1}\right) = \quad — // — \quad +$ no base pts.

Simple space, much simpler than

for fixed $\mathcal{L}$, $\mathbb{P}(H^0(C,\mathcal{L})^n) = H^0(C,\mathcal{L})^n / \mathrm{Aut}(\mathcal{L}) = \mathbb{C}^*$.

$$\mathrm{QM}\left(C \xrightarrow{f} [Y/G]\right) = \left\{ \begin{array}{l} \mathcal{P} = \text{a principal } G \text{ bundle over } C \\ f = \text{a section of } \mathcal{P}\times_G Y \end{array} \right\}$$

$$\begin{array}{c} \mathcal{P}\times_G Y \\ \downarrow \uparrow f \\ C \end{array}$$

base locus $= f^{-1}(Y_{\text{unstable}})$

0-dimensional and disjoint from nodes and maybe some other marked pts of C

What is this for $\mathrm{Hilb}(\mathbb{C}^2, n)$?

$$W = \mathbb{C} \underset{B}{\overset{A}{\rightleftarrows}} V \circlearrowleft x_1, x_2 \qquad \mu = [x_1, x_2] + AB \subset \mathfrak{gl}(V)^*$$

$$\mu^{-1}(0)/GL(V)$$

stable locus $x_1^i x_2^j A \cdot 1$ span $V \implies B = 0$

$\boxed{\mathrm{QM}}$

$$A \quad V \circlearrowleft x_1, x_2 \qquad [x_1, x_2] + AB = 0$$

$\boxed{\text{QM}}$

$$\mathcal{W} = \mathcal{O}_C \underset{B}{\overset{A}{\rightleftarrows}} \mathcal{V} \circlearrowleft x_1, x_2$$

$\mu = [x_1, x_2] + AB = 0$

$B = 0.$

$\mathcal{W}$: fixed, acted upon by thing like $GL(W)$

$\mathcal{V}$: a vector bundle over C (varies in moduli)

$x_i \in \mathrm{Hom}_C^{\cdot}(\mathcal{V}, \mathcal{V})$.

a base point is where $x_1^i x_2^j A$ fail to generate the fiber of $\mathcal{V}$

$\mathcal{V}$ + 2 operators x_1, x_2 on it $\Longleftrightarrow$ sheaf $\mathcal{F}$ on $C \times \mathbb{C}^2$

bundle on C

pure of dimension 1.

$\mathcal{V}$ = pushforward $(\mathcal{F})$ to C

with coordinates x_1, x_2.

$$\mathcal{O}_{C \times \mathbb{C}^2} \longrightarrow \mathcal{F}$$

image of $x_1^i x_2^j A\, \mathcal{O}_C$

base point = support of cokernel

$$\mathrm{QM}(C \to \mathrm{Hilb}(\mathbb{C}^2, n)) = \mathrm{PT}(C \times \mathbb{C}^2) = \{ \mathcal{O}_{\text{3-fold}} \longrightarrow \mathcal{F} \text{ of pure dim } 1,\ \dim \mathrm{Coker} = 0 \}$$

Pandharipande Thomas

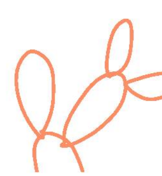
good for any 3-fold

Variation: $\mathbb{C}^2$ bundle over C instead of $C \times \mathbb{C}^2$ $\Leftarrow$ Twist the quiver data by $\mathrm{Aut}(X) \supset GL(W) \times GL(\text{edge}) \times \mathbb{C}^\times_\hbar$.

$$\mathcal{L}_1 \oplus \mathcal{L}_2 \to C$$

$x_i \in \mathrm{Hom}(\mathcal{V}, \mathcal{V} \otimes \text{line bundle on } C)$

line bundle on C $= \mathcal{L}_i^\vee$



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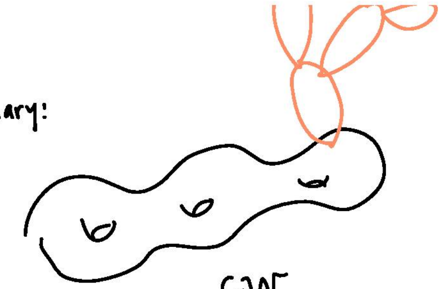
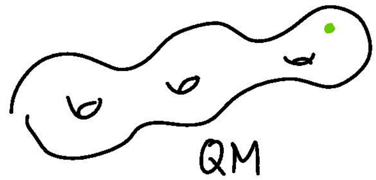
Summary:



GW

$X \approx$ arbitrary smooth
+ e.g. projective over affine

$\mathcal{L}_i^\vee$



base pt

does depend on the ambient stack $\mathfrak{X}$

QM

For properness

$$X \underset{\text{stable}}{\subset} \left[{}^{\text{affine}} / {}_{\text{reductive}} \right]$$

↳ elementary discussion in Park City notes

For obstruction theory

locally complete intersection.

$\mu^{-1}(0)$ is O.K.

(the point of having B is to have # of equation = codim)

$$\left[\underset{\text{on smooth}}{\mathrm{Crit}(\text{function})} / {}_{\text{reductive}} \right] \supset$$

some scaling properties

$\xi \in \mathfrak{g}$ ← extra variable

function $= \langle \xi, \mu(x) \rangle$

$\mathrm{Crit} = \{ \mu(x) = 0,\ \xi \cdot x = 0 \}$

+

stability $\Longrightarrow$ $\mu = 0, \xi = 0.$

$\mathrm{QM}(C \to \mathrm{Hilb}(\text{ADE surface})) =$ Bryan-Steinberg

read Henry Liu "Quasimaps and stable pairs"