Virtual Fundamental Classes in Enumerative Geometry

Tue. 9/29/2020

Enumerative problem

$\rightsquigarrow$ moduli space $\mathcal{X}$ $\leftarrow$ virtually smooth DM stack

$\quad\quad$ perfect obstruction theory $E \in Ob(D(\mathcal{X}))$

$$E = [E^{-1} \to E^0]$$

$$T^{vir} = (E^0)^\vee - (E^{-1})^\vee$$

$$\text{vrdim } E = \text{rank } T^{vir} = rk\, E^0 - rk\, E^{-1} = d$$

Behrend – Fantechi

$(\mathcal{X}, E) \longrightarrow$ virtual fundamental class (VFC)

$$[\mathcal{X}, E]^{vir} \in A_d(\mathcal{X}; \mathbb{Q}) =: A_d(\mathcal{X})$$

- If $\mathcal{X}$ is a scheme $\quad [\mathcal{X}, E]^{vir} \in A_d(\mathcal{X}; \mathbb{Z})$
- If $\mathcal{X}$ is smooth $\quad T^{vir} = T_{\mathcal{X}} \iff E = [0 \to \Omega_{\mathcal{X}}]$

  $d = \dim \mathcal{X} \quad\quad [\mathcal{X}, E]^{vir} = [\mathcal{X}] \in A_d(\mathcal{X}).$

<u>Y. P. Lee</u> $(\mathcal{X}, E) \rightsquigarrow$ virtual structure sheaf

$$\mathcal{O}^{vir}_{\mathcal{X},E} \in D(\mathcal{X})$$

If $\mathcal{X}$ is smooth, and $T^{vir} = T_{\mathcal{X}} \Leftrightarrow E = [0 \to \Omega_{\mathcal{X}}]$

then $\mathcal{O}^{vir}_{\mathcal{X},E} = \mathcal{O}_{\mathcal{X}}$

$\mathcal{X}$ DM stack

A perfect obstruction theory on $\mathcal{X}$ is

$E \in ob(D^{[-1,0]}(\mathcal{X})) \subset ob(D(\mathcal{X}))$

$h^i(E) = 0$ unless $i = 0, -1$

$\phi : E \to L_{\mathcal{X}}$

cotangent complex of $\mathcal{X}$

$E = [\cdots \to E^{-1} \to E^0]$ $\quad h^0(\phi)$ isomorphism

$\downarrow \phi$

$L_{\mathcal{X}} = [\cdots \to L^{-1}_{\mathcal{X}} \to L^0_{\mathcal{X}}]$ $\quad h^{-1}(\phi)$ surjection

$\mathcal{X}$ is scheme $\quad \coprod_{\alpha} U_\alpha \to \mathcal{X}$

$\operatorname{Spec} R_\alpha = U_\alpha$ affine scheme

$U_\alpha \to \mathcal{X}$ open embedding

$\mathcal{X}$ is a DM stack $\coprod_\alpha U_\alpha \to \mathcal{X}$ atlas
[an Artin stack]

U_α affine scheme $U_\alpha \to \mathcal{X}$ étale
[smooth]

$$\begin{array}{ll} U \text{ affine} & \boxed{\phi_U : E_U \to L_U} \\ \downarrow \text{étale} & \\ \mathcal{X} & \phi : E \to L_{\mathcal{X}} \end{array}$$

$U \xrightarrow{i} W$ i closed embedding
W nonsingular scheme
I = Ideal sheaf of U in W

Fulton normal cone $C_{U/W} = \mathrm{Spec}\left(\bigoplus_{n\geq 0} I^n/I^{n+1}\right)$
normal sheaf $N_{U/W} = \mathrm{Spec}\left(\mathrm{Sym}^\bullet(I/I^2)\right)$

$$\begin{array}{ccccc} E_U = & [\, E_U^{-1} & \to & E_U^0 \,] \\ & \downarrow & & \wr\wr \\ \tau_{\geq -1} L_U = & [\, I/I^2 & \to & i^*\Omega_W \,] \end{array}$$

$(E)_U = \mathrm{Spec}(\mathrm{Sym}^\bullet E^{-1})$
$= (E_U^{-1})^\vee$

$$i^* T_W \subseteq (E_U^0)^\vee$$

pure dim = dim W

$$C_{U/W} \subset N_{U/W} \subset (E_1)_U$$

cone over U — abelian cone over U — vector bundle of rank d_1 over U

$$[C_{U/W}/i^* T_W] \hookrightarrow [N_{U/W}/i^* T_W] \hookrightarrow [(E_1)_U/i^* T_W]$$

$$\underline{C}_U \qquad \underline{N}_U \qquad \underline{\mathcal{E}}_U$$

cone stack $/U$ — abelian cone stack $/U$ — vector bundle stack$/U$

$\underline{C}_U$ intrinsic normal cone pure dim 0

$\underline{N}_U$ sheaf

$\underline{\mathcal{E}}_U$

descends to $\mathfrak{X}$: $\underline{C}_{\mathfrak{X}} \subset \underline{N}_{\mathfrak{X}} \hookrightarrow \underline{\mathcal{E}}_{\mathfrak{X}}$

closed embedding of cone stacks over $\mathfrak{X}$

$\underline{C}_{\mathfrak{X}}$ intrinsic normal cone. cone stack over $\mathfrak{X}$

$\underline{N}_{\mathfrak{X}}$ intrinsic normal sheaf. abelian cone stack of $\mathfrak{X}$

$\underline{\mathcal{E}}_{\mathfrak{X}}$ vector bundle stack of rank $d_1 - d_0$ $= -\text{vdim}$

$$\begin{array}{ll} \underline{\mathcal{E}}_{\mathcal{X}} & \text{vector bundle stack} \\ \downarrow \pi & \\ \mathcal{X} & \text{DM stack} \end{array}$$

<u>Fulton</u>

$$\begin{array}{ll} E & \text{vector bundle of rank } r \\ \downarrow & \\ X & \text{scheme} \end{array}$$

Gysin map $\quad 0_E^! : A_d(E;\mathbb{Z}) \to A_{d-r}(X;\mathbb{Z})$

<u>Kresch</u>

$$\begin{array}{ll} \underline{\mathcal{E}} & \text{vector bundle stack of rank } r \\ \downarrow & \\ \mathcal{X} & \text{DM stack} \end{array}$$

Gysin map $\quad 0_E^! : A_d(\underline{\mathcal{E}}) \to A_{d-r}(\mathcal{X})$

$$[\mathcal{X}, E]^{vir} := 0_{\underline{\mathcal{E}}}^! [C_{\mathcal{X}}]^{vir} \in A_d(\mathcal{X})$$

$\underline{C}_{\mathcal{X}} \subset \underline{\mathcal{E}} \qquad [\underline{C}_{\mathcal{X}}] \in A_0(\underline{\mathcal{E}})$

pure dim 0 $\qquad$ rank $-d$

$$\begin{tikzcd} \underline{\mathcal{E}}_U \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \underline{\mathcal{E}}_{\mathcal{X}} \arrow[d] \\ U \arrow[r, "\text{étale}"] \arrow[d] & \mathcal{X} \\ \text{affine scheme} & \end{tikzcd}$$

$$\begin{tikzcd} E = [E_U^{-1} \arrow[r] & E_U^0] \\ \tau_{\geq -1} L_U = [I_W/I_W^2 \arrow[r] \arrow[u, leftarrow, "\phi"] & i^*\Omega_W] \arrow[u, "\simeq"'] \end{tikzcd}$$

$i : U \xrightarrow{\text{closed embedding}} W$ $\hookrightarrow$ nonsingular scheme

$$\begin{tikzcd} C_{U/W} \arrow[r, hook] \arrow[d] & N_{U/W} \arrow[r, hook] \arrow[d] & (E_1)_U = (E_U^{-1})^\vee \arrow[d, "\pi"] \\ \underline{C}_U \arrow[r, phantom, "\subset"] & \underline{N}_U \arrow[r, phantom, "\subset"] & \underline{\mathcal{E}}_U = [(E_1)_U / i^* T_W] \end{tikzcd}$$

π is smooth of relative dim $d_0 = \dim W = \operatorname{rank} E^0$
$d_1 = \operatorname{rank} E^{-1}$

virtual dim $d = d_0 - d_1 = -\operatorname{rank} \underline{\mathcal{E}}_U$

$$\begin{tikzcd} {[C_{U/W}]} \in A_{d_0}((E_1)_U) \arrow[r, "0^!_{E_1}"] & A_{d_0-d_1}(U) \\ {[\underline{C}_U]} \quad A_0(\underline{\mathcal{E}}_U) \arrow[u, "\pi^*"] \arrow[r, "0^!_{\underline{\mathcal{E}}_U}"] & A_d(U) \arrow[u, equal] \end{tikzcd}$$

VFC $\quad [U, E_U]^{vir} = 0^!_{(E_1)_U}[C_{U/W}] = 0^!_{\underline{\mathcal{E}}_U}[\underline{C}_U] \in A_d(U)$

virtual structure sheaf

$$\mathcal{O}^{vir}_{U,E_U} = \mathcal{O}_U \overset{der}{\underset{\mathcal{O}_{(E_1)_U}}{\otimes}} \mathcal{O}_{C_{U/W}} = \mathcal{O}_U \overset{der}{\underset{\mathcal{O}_{\underline{E}_U}}{\otimes}} \mathcal{O}_{\underline{C}_U} \in \mathrm{Ob}(D(U))$$

$$\mathcal{O}^{vir}_{\mathcal{X},E_{\mathcal{X}}} = \mathcal{O}_{\mathcal{X}} \overset{der}{\otimes} \mathcal{O}_{\underline{\mathcal{E}}_{\mathcal{X}}} \in \mathrm{Ob}(D(\mathcal{X}))$$

virtual structure sheaf on a virtually smooth DM stack $(\mathcal{X}, E_{\mathcal{X}})$

U smooth scheme

$i = id : U \to U$

$C_{U/U} = U$

$E_U = [0 \to \Omega^1_U]$

$\parallel$

$L_U = [0 \to \Omega^1_U]$

vector bundles

$E_0 \to E_1$

$\searrow \swarrow$

U

$$\begin{array}{ccl} C_{U/U} = U & & 0^! = id : A_d(U;\mathbb{Z}) \to A_d(U;\mathbb{Z}) \\ \downarrow & \downarrow \text{smooth relative dim } d = \dim U = rk\, T_U & \\ \underline{C}_U = \underline{E}_U = [U/T_U] & & \end{array}$$

E_1

$$[U, E_U]^{vir} = 0^! [C_{U/U}] = [U] \in A_d(U;\mathbb{Z})$$

$\mathcal{X}$ smooth DM stack $\qquad E_{\mathcal{X}} = [0 \to \Omega_{\mathcal{X}}] = L_{\mathcal{X}}$

$$\Rightarrow [\mathcal{X}, E_{\mathcal{X}}]^{vir} = [\mathcal{X}] \in A_d(\mathcal{X})$$

$$\mathcal{O}^{vir}_{U,E_U} = \mathcal{O}_U \otimes_{\mathcal{O}_U} \mathcal{O}_U = \mathcal{O}_U \quad , \quad \mathcal{O}^{vir}_{\mathcal{X},E_{\mathcal{X}}} = \mathcal{O}_{\mathcal{X}} \in \mathrm{Ob}(D(\mathcal{X}))$$

Equivariant Version

$$G \curvearrowright \mathcal{X} \xrightarrow{} \text{DM stack} \quad \Rightarrow \quad L_{\mathcal{X}} \in Ob(D_G(\mathcal{X}))$$

Romagny "Group action on stacks and applications"

$E \in Ob(D_G^{[-1,0]}(\mathcal{X}))$ G-equivariant perfect obstruction theory of vir dim d

$[\mathcal{X}, E]^{vir} \in A_d^G(\mathcal{X})$ $\qquad \mathcal{O}^{vir}_{\mathcal{X},E} \in D_G(\mathcal{X})$

$G = (\mathbb{C}^*)^k$ torus localization of VFC.

Graber-Pandharipande, Behrend, Chang-Kiem-Li

$$[\mathcal{X}, E]^{vir} = i_*\left(\frac{[\mathcal{X}^G, E^f]^{vir}}{e_T(N^{vir})}\right) \in A_*^G(\mathcal{X}) \otimes_{R_G} S_G$$

$i : \mathcal{X}^G \hookrightarrow \mathcal{X}$ $\qquad R_G = A_*^G(\bullet) = \mathbb{Q}[t_1, \ldots, t_k]$

$E|_{\mathcal{X}^G} = E^f \oplus E^m$ $\qquad$ point

$T^{vir}_{\mathcal{X}}|_{\mathcal{X}^G} = T^{vir}_{\mathcal{X}^G} \oplus N^{vir}$ $\qquad S_G = \mathbb{Q}(t_1, \ldots t_k)$

Relative version

$\mathcal{X}$ DM stack

$\pi_{\mathcal{X}/\mathcal{M}} : \mathcal{X} \to \mathcal{M}$

$\hookrightarrow$ smooth Artin stack of dim m

$\pi_{\mathcal{X}/\mathcal{M}}$ virtually smooth of relative dim d

i.e. $\exists$ relative perfect obstruction theory

$E_{\mathcal{X}/\mathcal{M}} \in Ob(D^{[-1,0]}(\mathcal{X}))$

$\phi : E_{\mathcal{X}/\mathcal{M}} \to L_{\mathcal{X}/\mathcal{B}}$ relative cotangent cpx

$h^0(\phi)$ isomorphism, $h^{-1}(\phi)$ surjection

$$\begin{tikzcd} U \arrow[r, "\text{étale}"] \arrow[dr] & \mathcal{X} \arrow[d] \\ & \mathcal{M} \end{tikzcd}$$

affine scheme $\to U$

$$\begin{tikzcd} U \arrow[r, "i"] \arrow[d] & W \arrow[dl] \\ \mathcal{M} & \end{tikzcd}$$

i closed embedding to a scheme

smooth of relative dim d_0

$\underline{C}_{U/\mathcal{M}} = [C_{U/W} / i^* T_{W/\mathcal{M}}]$ pure dim m

$\underline{N}_{U/\mathcal{M}} = [N_{U/W} / i^* T_{W/\mathcal{M}}]$

$\underline{C}_{U/\mathcal{M}} \subset N_{U/\mathcal{M}} \subset \underline{\mathcal{E}}_{U/\mathcal{M}} \to$ vector bundle of rank $-d$

$[U, E_{U/\mathcal{M}}]^{vir} = \underline{0}^!_{\underline{\mathcal{E}}_{U/\mathcal{M}}} \underbrace{[\underline{C}_{U/\mathcal{M}}]}_{A_m(\underline{E}_{U/\mathcal{M}})} \in A_{m+d}(U)$

$$[\mathfrak{X}, E_{\mathfrak{X}/\mathfrak{M}}]^{vir} \in 0^!_{E_{\mathfrak{X}/\mathfrak{M}}}[C_{\mathfrak{X}/\mathfrak{M}}] \in A_{*}(\mathfrak{X})$$

$$\mathcal{O}^{vir}_{\mathfrak{X}, E_{\mathfrak{X}/\mathfrak{M}}}$$

<u>Application</u> GW theory

X nonsingular projective variety

$\beta \in H_2(X,\mathbb{Z})$ effective curve class

$\hat{\mathfrak{X}} = \mathfrak{M}^{pre}_{g,n}(X,\beta)$ moduli of genus g, n-pointed, deg β prestable maps to X

$Ob(\hat{\mathfrak{X}}(\cdot))$ $\quad ((C, x_1 \dots x_n), f) \quad f: C \to X \quad f_*[C] = \beta$

$C, x_1 \dots x_n$: genus g n pointed prestable curve

$Mor(\hat{\mathfrak{X}}(\cdot))$

$$\begin{tikzcd} (C, x_1 \dots x_n) \arrow[d, "\wr\,\phi"'] \arrow[dr, "f"] \arrow[dr, phantom, "\circlearrowleft", shift right=3] & \\ (C', x_1' \dots x_n') \arrow[r, "f'"'] & X \end{tikzcd}$$

$\mathfrak{X} = \overline{M}_{g,n}(X,\beta) \subset \hat{\mathfrak{X}}$ Artin stack

stable map $|Aut| < \infty$

$\mathfrak{X}$: proper DM stack

$$\begin{array}{c} \widetilde{\mathfrak{X}} \\ \downarrow \pi_{\mathfrak{X}/\mathfrak{M}} \\ \mathfrak{M} = \mathfrak{M}_{g,n} \end{array}$$

virtually smooth

moduli of genus g n-pted prestable curves

smooth Artin stack of dim $3g-3+n$

$f: C \to X$

$$\begin{tikzcd} & X \\ C_{\widetilde{\mathfrak{X}}} \arrow[ur, "ev_{\mathfrak{X}}"] \arrow[r] \arrow[d, "\pi_{\mathfrak{X}}"'] \arrow[dr, phantom, "\square"] & C_{\mathfrak{M}} \arrow[d, "\pi_{\mathfrak{M}}"] \\ \widetilde{\mathfrak{X}} \arrow[r] & \mathfrak{M} \end{tikzcd}$$

universal curve

$\widetilde{\mathfrak{X}} = \Gamma(C_\mathfrak{M} \times X / C_\mathfrak{M})$

$\mathrm{Mor}_\beta(C,X) \longrightarrow \xi = (C, x_1 \dots x_n)$

$$E_{\widetilde{\mathfrak{X}}/\mathfrak{M}} = \left(R^\bullet \pi_{\mathfrak{X}*}\, ev_{\mathfrak{X}}^* T_X\right)^\vee$$

$h^i(E^\vee_{\widetilde{\mathfrak{X}}/\mathfrak{M}})|_\xi$ $\quad H^0(C, f^*T_X) \quad i=0$

$\quad H^1(C, f^*T_X) \quad i=1$

$h^i(T_\mathfrak{M})|_\xi$ $\quad \mathrm{Ext}^0_{O_C}(\Omega_C(D), O_C) \quad i=-1$

$\quad \mathrm{Ext}^1_{O_C}(\Omega_C(D), O_C) \quad i=0$

relative vir dim $\widetilde{\mathfrak{X}}/\mathfrak{M} = \chi(C, f^*T_X) = \int_\beta c_1(TX) + \dim X(1-g)$

$d^X_{g,n,\beta} := \mathrm{vir\,dim}\, \overline{M}_{g,n}(X,\beta) = \int_\beta c_1(TX) + (\dim X - 3)(1-g) + n$

$[\overline{M}_{g,n}(X,\beta)]^{vir} \in A_{d^X_{g,n,\beta}}(\overline{M}_{g,n}(X,\beta))$

Application 2 quasimap theory
cf. Ciocan-Fontanine, Kim, Maulik

X nonsingular (quasi) projective

$$X = W/\!\!/_{\theta} G = W^{ss}/G \quad \begin{tikzcd}[row sep=small] & {[W/G]} \arrow[d] \\ {} \arrow[ur] \arrow[r] & X_0 = W/\!\!/_{aff} G \end{tikzcd}$$

where

$W = \operatorname{Spec} A$ affine, at most l.c.i. singularities

G reductive complex algebraic group

$\theta \in \operatorname{Pic}_G(W)$

Assume $\begin{cases} W^s = W^{ss} \text{ is nonsingular} \\ G \text{ acts freely on } W^s \end{cases}$

$[W/G]$ quotient stack $\to$ Artin stack

$X_0 = X/\!\!/_{aff} G = \operatorname{Spec}(A^G)$ affine quotient

e.g. $X = T^*V /\!\!/\!\!/ G = \mu_{\mathbb{C}}^{-1}(0)/\!\!/G = \mu_{\mathbb{C}}(0)^{ss}/G$

algebraic symplectic reduction

maps to X

$$\begin{tikzcd} C \arrow[r, "f"] & X = W^{ss}/G \arrow[d] \\ & {[\cdot/G] = BG} \end{tikzcd}$$

$\longleftrightarrow$

$$\begin{tikzcd} P \arrow[r, "\tilde{f}"] \arrow[d] & W^{ss} \\ C & \end{tikzcd}$$

$P \to C$ principal G-bundle

$\tilde{f} : P \to W^{ss}$ G-equivariant

$\updownarrow$

$$\begin{tikzcd} P \times_G W^{ss} \arrow[d] \\ C \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

$P \to C$ principal G-bundle

$\sigma : C \to P \times_G W^{ss}$ section of the fiber bundle $P \times_G W^{ss} \to C$ with fiber W^{ss}

(prestable) quasimap to X

$$\begin{tikzcd} C \arrow[r, "f"] & {[W/G]} \arrow[d] \\ & {[\bullet/G] = BG} \end{tikzcd}$$

$B \subset C$ finite

$f(C \setminus B) \subset X$

$$\longleftrightarrow$$

$$\begin{tikzcd} P \arrow[r, "\tilde{f}"] \arrow[d, "\pi"] & W \\ C \end{tikzcd}$$

$B \subset C$ finite

$\tilde{f}(\pi^{-1}(C \setminus B)) \subset W^{ss}$

$P \to C$ principal G-bundle

$\tilde{f} : P \to W$ G-equivariant

$$\updownarrow$$

$$\begin{tikzcd} P \times_G W \arrow[d] \\ C \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

$B \subset C$ finite

$\sigma(C \setminus B) \subset P \times_G W^{ss}$

$P \to C$ principal G-bundle

$\sigma : C \to P \times_G W$ section of the fiber bundle $P \times_G W$ with fiber W

$\widetilde{\mathfrak{X}}$ moduli of genus g, n-pointed, deg β prestable quasimaps to $X = W /\!/_\theta G$

$\beta \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}^G(W), \mathbb{Z})$

$\mathrm{Ob}(\widetilde{\mathfrak{X}}(\cdot))$ $\quad ((C, x_1, \dots, x_n), P, \sigma)$

genus g n-pointed prestable curve

G-equivariant

$P \xrightarrow{\tilde{f}} W$ $\longleftrightarrow$ $P \to C$ principal G-bundle

$L \in \mathrm{Pic}^G(W)$ $\quad \sigma : C \to P \times_G W$ section

$\tilde{f}^* L \in \mathrm{Pic}^G(P) = \mathrm{Pic}(C)$ $\quad B = \sigma^{-1}(P \times_G (W \setminus W^{ss}))$ is finite

$\deg \tilde{f}^* L = \beta(L)$

$\mathrm{Mor}(\widetilde{\mathfrak{X}}(\cdot))$

$(C, x_1, \dots, x_n), P, \sigma) \to ((C', x_1', \dots, x_n'), P', \sigma')$

$\phi_1 : (C, x_1, \dots, x_n) \xrightarrow{\cong} (C', x_1', \dots, x_n')$

$\phi_2 : P \xrightarrow{\cong} \phi_1^* P'$ $\qquad P \times_G W \xrightarrow{\cong} \phi_1^*(P' \times_G W)$

$$\begin{tikzcd} P\times_G W \arrow[rr, "\phi_2", "\cong"'] \arrow[dr] & & \phi_1^*(P'\times_G W) \arrow[dl] \\ & C \arrow[ul, bend left, "\sigma"] \arrow[ur, bend right, "\phi_1^*\sigma'"'] & \end{tikzcd}$$

DM stack

$$\mathcal{X} = Qmap^{\epsilon}_{g,n}(W/\!\!/_\theta G, \beta)$$

ϵ-stable quasimap

$\epsilon = 0^+$ stable quasimap

$\epsilon = +\infty$ stable map

$$\cap$$

$$\sim$$

Artin stack $\to$ $\tilde{\mathcal{X}} \subset \Gamma(P_{\tilde{\mathcal{X}}}\times_G W / C_m)$

$$\downarrow \pi_{\tilde{\mathcal{X}}/\mathcal{B}}$$

$$\mathcal{B} \subset Bun^G_{g,n} = \Gamma(C_m \times BG / C_m)$$

smooth Artin stack

$$\downarrow \pi_{\mathcal{B}/\mathfrak{M}}$$

$\mathfrak{M} = \mathfrak{M}_{g,n}$ smooth Artin stack of dimension $\boxed{3g-3+n}$ Φ

$$E_{\mathfrak{M}} = L_{\mathfrak{M}}$$

$$E_{\mathfrak{M}}^{\vee} = T_{\mathfrak{M}}$$

$$\xi = (C, x_1, \dots, x_n)$$

$$h^i(T_{\mathfrak{M}})\big|_{\xi} = \begin{cases} Ext^0_{\mathcal{O}_C}(\Omega_C(D), \mathcal{O}_C), & i=-1 \\ Ext^1_{\mathcal{O}_C}(\Omega_C(D), \mathcal{O}_C), & i=0 \\ 0 & ,\text{otherwise} \end{cases}$$

$E_{\mathcal{B}/\mathcal{M}} = L_{\mathcal{B}/\mathcal{M}}$ $\quad$ $\xi = (C, x_1, \dots, x_n, P)$

$E^\vee_{\mathcal{B}/\mathcal{M}} = T_{\mathcal{B}/\mathcal{M}}$

$$C \xrightarrow{P} BG$$

$$T_{BG} = \mathfrak{g}[-1]$$

$\mathfrak{g}$ adjoint representation of G

$$h^i(T_{\mathcal{B}/\mathcal{M}})\big|_\xi = H^i(C, P^* T_{BG})$$
$$= H^i(C, P \times_G \mathfrak{g})[-1]$$

$$= \begin{cases} H^0(C, P \times_G \mathfrak{g}), & i = -1 \\ H^1(C, P \times_G \mathfrak{g}), & i = 0 \\ 0, & \text{otherwise} \end{cases}$$

$\pi_{\mathcal{B}/\mathcal{M}} : \mathcal{B} \to \mathcal{M}$ smooth of relative dim $\boxed{\dim G \cdot (g-1)}$ 2

$$\begin{tikzcd} P_{\mathcal{B}} \arrow[d] & \\ C_{\mathcal{B}} \arrow[r] \arrow[d, "\pi_{\mathcal{B}}"'] \arrow[dr, phantom, "\square"] & C_{\mathcal{M}} \arrow[d, "\pi_{\mathcal{M}}"] \\ \mathcal{B} \arrow[r] & \mathcal{M} \end{tikzcd}$$

$P_{\mathcal{B}}$ universal principal G-bundle

$C_{\mathcal{M}}$ ← universal curve

$$h^i(E^\vee_{\mathcal{B}/\mathcal{M}}) = R^{i+1}\pi_{\mathcal{B}*}(P_{\mathcal{B}} \times_G \mathfrak{g})$$

$$\xi = ((C, x_1, \ldots, x_n), P, \sigma)$$

$$\begin{array}{c} P \times_G W \\ \downarrow \uparrow \sigma \\ C \end{array}$$

Special Case: W is nonsingular

$$h^i(E^\vee_{\tilde{\mathcal{X}}/\mathcal{B}})\Big|_\xi = H^i(C, \sigma^* T_{P\times_G W/C})$$

$$= \begin{cases} H^0(C, \sigma^* T_{P\times_G W/C}) & i=0 \\ H^1(C, \sigma^* T_{P\times_G W/C}) & i=1 \\ 0 & \text{otherwise} \end{cases}$$

relative virtual dim of $\pi_{\tilde{\mathcal{X}}/\mathcal{B}}$

$$\deg(\sigma^* T_{P\times_G W/C}) + \operatorname{rank}(\sigma^* T_{P\times_G W/C})(1-g)$$

$\det(T_W) \in \operatorname{Pic}_G(W)$

$$= \boxed{\beta(\det(T_W)) + \dim W\,(1-g)} \quad ③$$

$$\text{vir.dim. } \tilde{\mathcal{X}} = \beta(\det(T_W)) + (\dim W - \dim G)(1-g) + 3g-3+n$$

①+②+③

$$= \beta(\det(T_W)) + (\dim X - 3)(1-g) + n$$

$$\begin{tikzcd}
P_{\tilde{\mathcal{X}}} \arrow[r] \arrow[d] & P_{\mathcal{B}} \arrow[d] & \\
C_{\tilde{\mathcal{X}}} \arrow[r] \arrow[d, "\pi_{\tilde{\mathcal{X}}}"] \arrow[dr, phantom, "\square"] & C_{\mathcal{B}} \arrow[r] \arrow[d, "\pi_{\mathcal{B}}"] \arrow[dr, phantom, "\square"] & C_{\mathcal{M}} \arrow[d, "\pi_{\mathcal{M}}"] \\
\tilde{\mathcal{X}} \arrow[r] & \mathcal{B} \arrow[r] & \mathcal{M}
\end{tikzcd}
\qquad
\begin{tikzcd}
P_{\tilde{\mathcal{X}}} \times_G W \arrow[d] \\
C_{\tilde{\mathcal{X}}} \arrow[u, bend right, "\sigma_{\tilde{\mathcal{X}}}"']
\end{tikzcd}$$

$$h^i(E^\vee_{\tilde{\mathcal{X}}/\mathcal{B}}) = R^i \pi_{\tilde{\mathcal{X}}*}(\sigma^*_{\tilde{\mathcal{X}}} T_{P_{\tilde{\mathcal{X}}} \times_G W / C_{\tilde{\mathcal{X}}}})$$