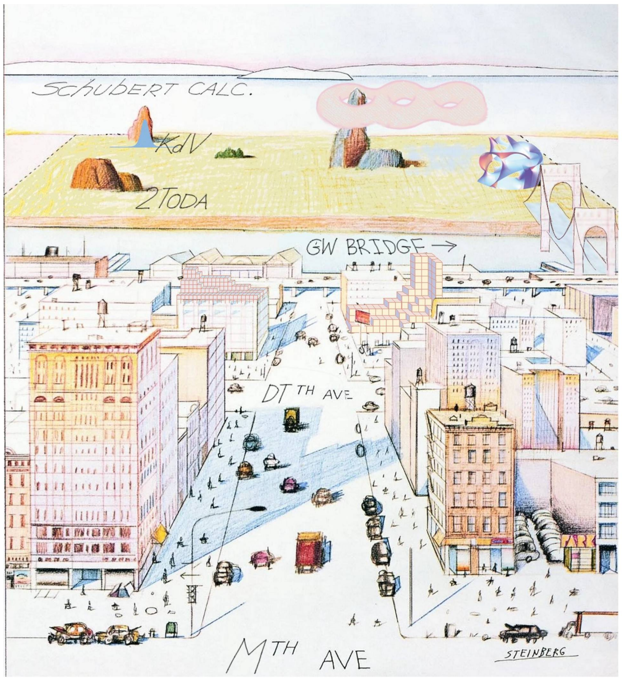
# Lecture 19, 09/22

Monday, September 21, 2020 6:32 PM




Enumerative geometry & geometric representation theory

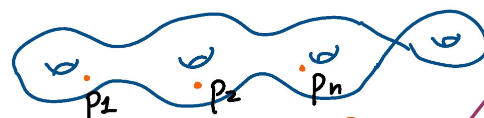
Start time Moscow 17:30
New York 10:30

The view of enumerative geometry* from near Columbia campus

* ≈ intersection theory on moduli spaces of geometric objects / susy states

Saul Steinberg (1914 – 1999) artist,
not Robert Steinberg (1922 – 2014) mathematician

$\overline{\mathcal{M}}_{g,n}$ = Deligne - Mumford moduli space




$g=4$, $n=3$

/ $\sim$

$T_{p_i}C$ = line bundle over

$$\left\langle \prod \tau_{k_i} \right\rangle = \int \prod c_1(T^*_{p_i} C)^{k_i}$$

$\dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n} = 3g-3+n = \sum k_i$

connected or <u>disconnected</u>

disconnected = exp(connected)

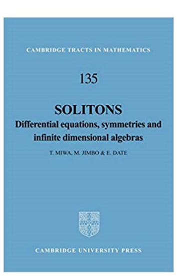
$\frac{1}{n!} = \frac{1}{|S(n)|}$

Witten: $\left\langle \exp\left(\sum_{k=0}^{\infty} \tau_k t_k\right) \right\rangle$ = tau function of KdV

Kontsevich, Mirzakhani, ....

KdV ⟵ KP equations ⟵ 2-Toda equations

with central ext.




⟶ this is about the action of $GL(\infty)$ in Fock module $\subset \Lambda^{\frac{\infty}{2}} \mathbb{C}^\infty$

<u>Rmk.</u> $\mathcal{U}(\mathfrak{gl}(\infty)) = Y(\widehat{\mathfrak{gl}(1)})$ at $\hbar = 0$

with central extension

$d_n = \begin{pmatrix} \ddots & & 0 \\ & \ddots & \\ 0 & & \ddots \end{pmatrix}$

infinite Töplitz matrices

$$\Lambda^{\frac{\infty}{2}}\mathbb{C}^\infty = \bigoplus_{n\in\mathbb{Z}} \mathcal{U}(\mathfrak{gl}(\infty))|n\rangle \qquad |n\rangle = e_n \wedge e_{n-1} \wedge e_{n-2} \wedge \ldots\ldots$$

basis of $\mathbb{C}^\infty$

$$\cong \quad \mathbb{C}[\alpha_{-1}, \alpha_{-2}, \ldots]|n\rangle$$

bilinear eq. on matrix elements of $g$

Image of $GL(\infty)$ cut out by equations

$$[g \otimes g, \text{Screening at } t_i = 0] = 0$$

Plücker relations among minors of $g$ $\Longrightarrow$

$$\| \quad \sum_i \psi_i \otimes \psi_i^* \qquad \psi_i = e_i \wedge$$

<u>2-Toda:</u>

initial cond.

$$\tau_n(x,y) = \langle n| \underbrace{\exp\left(\sum_{k>0} \frac{\alpha_k}{k} x_k\right) g \exp\left(\sum_{k<0} \frac{\alpha_k}{k} y_k\right)}_{\in GL(\infty)} |n\rangle$$

Take partials w.r.t. $x, y$ $\Longrightarrow$ get all possible matrix elements of

Plücker relations $\Longrightarrow$ 2 Toda PDE

More general principle: classically integrable $= \langle \ldots \exp \sum A_k t_k \ldots \rangle$

quantum integrable

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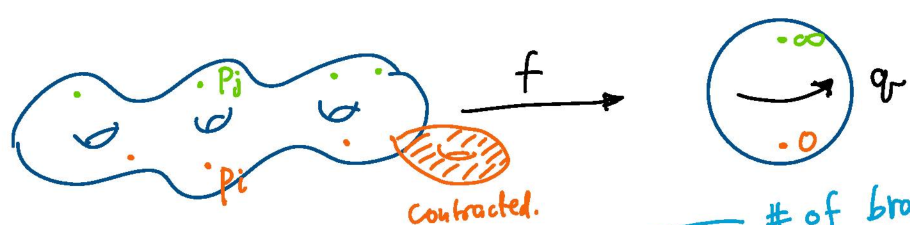
may be seen directly in GW of $\mathbb{P}^1$

$\varepsilon = \ln q \in \mathrm{Lie}\, \mathbb{C}^*$

$\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1)$ = moduli




$$\text{vir dim } \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1) = \boxed{2g + 2\deg - 2} + n$$

# of branch points of generic $f$

Euler char $= 2-2g$

Can talk abut

[O - Pandharipande]

Can talk abot

$$\left\langle \prod \tau_{k_i}(0) \prod \tau_{\ell_i}(\infty)\, Q^{\deg f}\, u^{-\chi} \right\rangle_{\mathbb{P}^1} =$$

$\chi$ = Euler char $= 2-2g$

[O.-Pandharipande]

$$= \left\langle 0 \right| \prod \tau_{k_i}(0)\, e^{\alpha_1} \Big(\underbrace{\frac{Q}{u^2}}_{\text{genus}}\Big)^{L_0} e^{\alpha_{-1}} \prod \tau_{\ell_i}(0)^* \left| 0 \right\rangle \quad \text{where}$$

$$= \langle 0 |\, \text{Toda}\; W^{-1} \dots W\; \text{Toda}\, | 0 \rangle$$

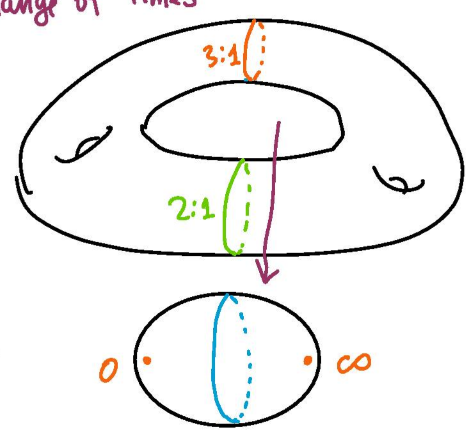
- $\tau_k(0) \in \mathfrak{gl}(\infty)$ explicit (depend on $u$)
- commute
- $\tau_k(0) = W\left( \frac{u^k}{(k+1)!}\alpha_{k+1} + \text{explicit linear}(\alpha_k, \dots, \alpha_1) \right) W^{-1}$

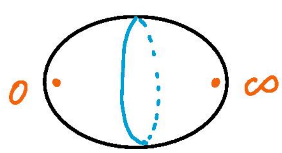
explicit and triangular

change of times

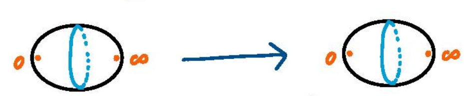
## Target space point of view

(3,2) = a partition of deg

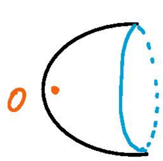



Fock space = space of states over 

$d$ copies of 

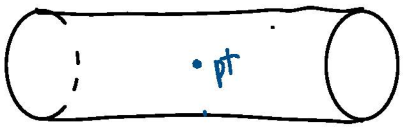
$$Q^d u^{-2d} = \left(\frac{Q}{u^2}\right)^{L_0}$$

 $= \left\langle \prod \tau_{k_i}(0)\, e^{\alpha_1} \right.$

Hodge integrals on GW side.

Nonequivariant theory = a certain limit = usual Toda for $\tau_k(pt)$ + Dubrovin–Zhang for $\tau_k(1)$, $1 = \frac{0 - \infty}{\varepsilon}$

$$\left\langle \prod \tau_{k_i}(pt)\, Q^{\deg} u^{\chi} \right\rangle = \langle 0 | e^{\alpha_1} \underset{\substack{\text{diagonal} \\ \text{matrices in} \\ \mathfrak{gl}(\infty)}}{\;} \left(\frac{Q}{u^2}\right)^{L_0} e^{\alpha_{-1}} | 0 \rangle$$




$\tau_k(pt)$ is an operator in Fock.

commute = diagonal matrices in $\mathfrak{gl}(\infty)$

commutative algebras in $Y(\widehat{\mathfrak{gl}}(1))$ of Baxter type

diagonal in $\Lambda^{\ell_i}$ = Schur functions

$$\langle 0 | e^{\alpha_1} \underset{\substack{\text{diagonal} \\ \text{matrices in} \\ \mathfrak{gl}(\infty)}}{\;} \left(\frac{Q}{u^2}\right)^{L_0} e^{\alpha_{-1}} | 0 \rangle = \sum_{\lambda} \left(\frac{\dim \lambda}{|\lambda|}\right)^2 \text{symmetric poly}(\lambda_i - i)$$

by definition

creates a partition

from creation and annihilation (multinomial)$^2$

a finite analog of $\displaystyle\int_{x \in \mathbb{R}^n} \prod (x_i - x_j)^2 P(x) e^{-\Sigma x_i^2}$

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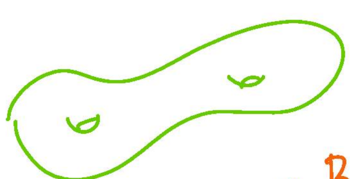
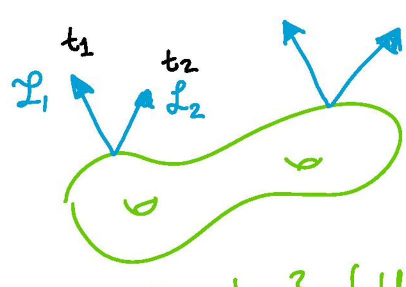
Natural to ask:
- is there a deeper reason we see partitions here
- what about $\hbar \neq 0$, i.e. $Y(\widehat{\mathfrak{gl}}(1))$ in its full glory

Answer in GW = DT theory of local curves in 3-fold.



Target curve $B$

$\xleftarrow[\text{if } \mathcal{L}_2 = \mathcal{L}_1^{\vee} \text{ and } t_2 = -t_1]{\text{Mumford}}$

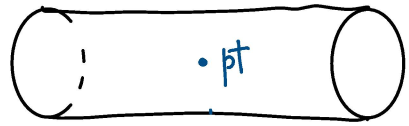



Target 3-fold. $X = \begin{array}{c} \mathcal{L}_1 \oplus \mathcal{L}_2 \\ \downarrow \\ B \end{array}$

$\hbar = -t_1 - t_2$

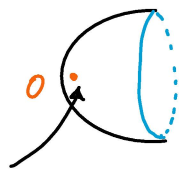
$GL(\infty)$

General $Y(\widehat{\mathfrak{gl}(1)})$




= Baxter subalgebra with $z = e^{iu}$

has to do with solving the quantum diff. eq. $\longrightarrow$

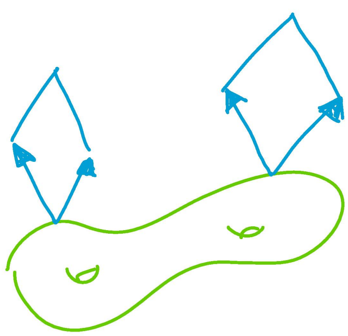



$\exp\left(\sum \tau_k(0)\, t_k\right)$

Poly in $\tau_k(0)$ $\longrightarrow$ Fock

understood in same terms, will explain later in the course

DT theory = enumerative theory of sheaves on 3-folds (and also objects in related categories)



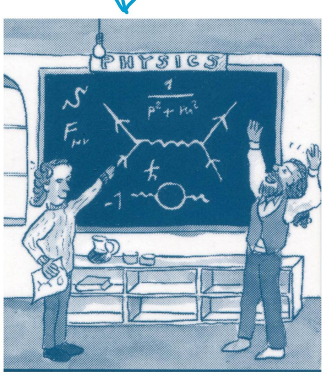
$$\begin{array}{ccc} S & \to & X \\ \text{surface} & & \downarrow \\ & & B \\ & & \text{curve} \end{array}$$

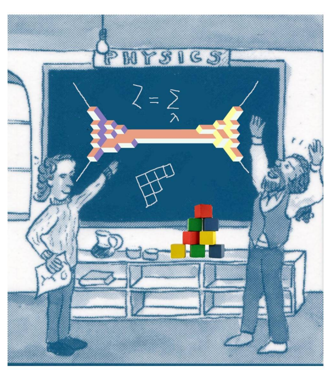
sheaf on $X$ $\approx$ map $B \longrightarrow$ moduli of sheaves on $S$

really a map if ~~flat~~ over $B$

otherwise a map with singularities

by Robbert Dijkgraaf



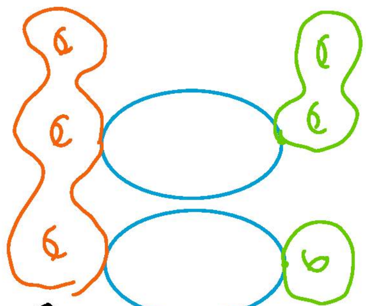




$\longleftarrow$ a monomial ideal sheaf on $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$

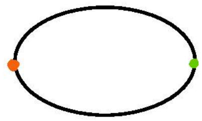
$$\downarrow$$

$$\mathbb{P}^1$$

has degree 2 over $\mathbb{P}^1$



Hodge integrals over



corresponds $\mathbb{P}^1 \dashrightarrow \mathrm{Hilb}(\mathbb{C}^2, 2)$

outside of $0, \infty$ $\longrightarrow$ 


$\longleftrightarrow$ Boxcounting (especially simple for PT counts).

Both solved explicitly in the language of quantum groups

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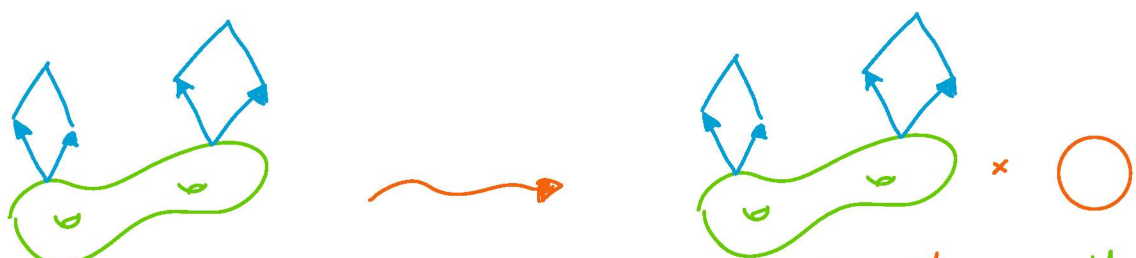
So far, we discussed Yangians, what about other quantum groups?

$U_q(\widehat{\widehat{gl}}(1)) \rightsquigarrow$ K-theory of DT Moduli spaces $X$ = local curve.



B

circle of time

$B \times S^1$ = world volume of M2 brane of M-theory

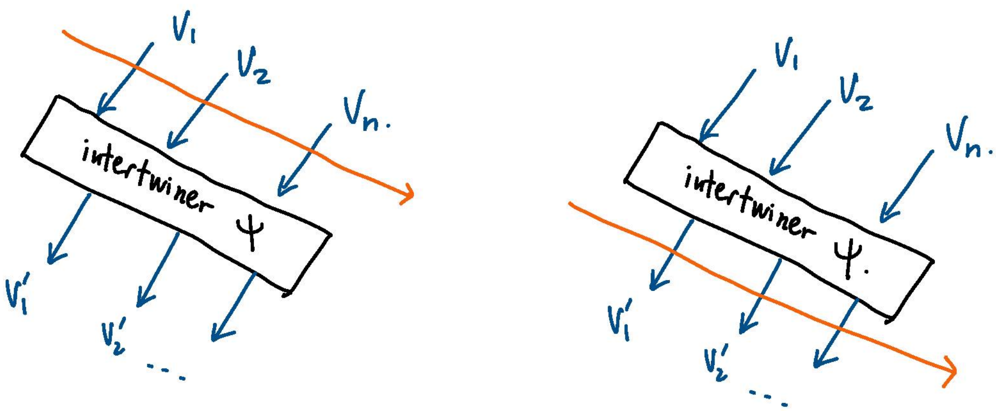
The $U_q(\widehat{\widehat{gl}}(1))$-theory is the theory of stack of M2-branes on $B \times S^1$ inside $Z \times S^1$.

$Z$ ← CY-5

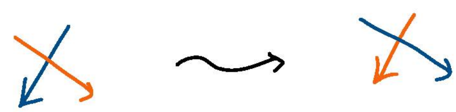
# of membranes = $L_0$

One of the many motivations to study 2+1 dimensional theories like we do in this course is to better understand M2 branes

Plücker ⟵ Screening ⟵ Slices




$\Psi$ commutes with quantum group



↪ a relation in quantum group.