\[ \overline{M_{g,n}} = \text{Deligne-Mumford moduli space} \]

\[ T_{p_i}C = \text{line bundle over} \]

\[ \left< \prod T_{p_i} \right> = \int T_{p_1} \cdots T_{p_n} \]

\[ \text{connected or disconnected} \]

\[ \text{disconnected} = \exp(\text{connected}) \]

\[ \dim \overline{M_{g,n}} = 3g - 3 + n = \sum k_i \]

Wilson: \[ \left< \exp \left( \sum_{k=0} T_k t_k \right) \right> = \text{tau function of KdV} \]

\[ \text{KdV} \leftarrow \text{KP equations} \leftarrow \text{2-Toda equations} \]

\[ \text{This is about the action of } GL(\infty) \text{ in Fock module } \wedge^\infty \mathcal{C} \]

\[ \text{Rmk. } \psi(\mathcal{O}) = Y(\mathcal{O}) \text{ at } t = 0 \]

\[ d_n = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \text{ infinite Toeplitz matrices} \]
\[ \bigoplus_{n \in \mathbb{Z}} U(\mathfrak{gl}(\omega)|n> = \text{basis of } \mathbb{C}^\omega \]

Image of GL(\omega) cut out by equations

\[ [g \otimes g, \text{Screening at } t = 0] = 0 \]

\[ \sum_i \psi_i \otimes \psi_i^* \]

\[ \psi_i = e_i \uparrow \]

2-Toda:

\[ \tau_n(x, y) = \langle n| \exp \left( \sum_{k > 0} \frac{b_k}{k} \alpha_k \right) g \exp \left( \sum_{k < 0} \frac{b_k}{k} y_k \right) |n> \]

Take partials w.r.t. \( x, y \) \[ \Rightarrow \text{get all possible matrix elements of } \in \text{GL}(\omega) \]

\[ \text{Plücker relations } \Rightarrow 2 \text{ Toda PDE} \]

More general principle: Classically integrable = \[ \langle \ldots \exp \sum \alpha_k t_k \ldots \rangle \]

quantum integrable

\[ \varepsilon = \ln q, \in \text{Lie } \mathbb{C}^* \]

may be seen directly in GW of \( \mathbb{P}^1 \)

\[ M_{g,n}(\mathbb{P}^1) = \text{moduli} \]

vir dim \( M_{g,n}(\mathbb{P}^1) = 2g + 2 \deg - 2 + n \)

\[ f \rightarrow g \]

\[ \text{Euler char} = 2 - 2g \]

\[ \text{Can talk about} \]

\[ \text{E. M. Assaad} \]
Can talk about
\[ \langle \prod \tau_{k_i}(0) \prod \tau_{\ell_i}(0) Q^{\deg f - \gamma} \rangle_{\mathbb{P}^1} = \left\langle \frac{Q}{u^2} \right\rangle_{\mathbb{P}^1}^{\alpha_1} \left( \prod \tau_{\ell_i}(0)^* \right) \left| \theta \right. \right. \]

Where
\[ \left\langle \frac{Q}{u^2} \right\rangle_{\mathbb{P}^1}^{\alpha_1} \]

- \( \tau_k(0) \in \mathfrak{g}_k(0) \) explicit (depend on \( u \))
- commute
- \( \tau_k(0) = W \left( \frac{u^k}{(k+1)!} \alpha_{k+1} + \text{explicit linear} \left( \alpha_k, \ldots, \alpha_1 \right) \right) W^{-1} \)

Explicit and triangular

Change of times

Target space point of view

(3,2) = a portion of deg

Fock space = space of states over

\[ d \text{ copies of } Q^d u^{2d} = \left( \frac{Q}{u^2} \right)^{\alpha_1} \]

Hodge integrals on GW side.

Nonequivariant theory = a certain limit = usual Toda for + Dubrovin-Zhang

For \( \tau_k(1), \theta = \frac{0 - \infty}{\varepsilon} \)
\[
\langle T_k(\mathbb{P}^1) \cdots T_1(\mathbb{P}^1) Q^x e^u \rangle = \langle 0 | e^{d_+} \text{ diagonal matrices in } \mathcal{O}_k(\mathbb{C}) e^{d_-} | 0 \rangle
\]

is an operator in Fock, commute = diagonal matrices in \( \mathcal{O}_k(\mathbb{C}) \)

\[
\sum_{\lambda} \left( \frac{\dim \lambda}{1 \lambda 1} \right)^2 \text{ symmetric poly } (\lambda_i - i)
\]

by definition

creates a partition

\[
\langle 0 | e^u \text{ diagonal matrices in } \mathcal{O}_k(\mathbb{C}) | 0 \rangle
\]

diagonal in \( \wedge e^u \) = Schur functions

\[
\text{commutative algebras in } \mathcal{Y}(\mathfrak{g}(\mathfrak{t})) \text{ of Baxter type}
\]

Natural to ask:

- is there a deeper reason we see partitions here
- what about \( \hbar \neq 0 \), i.e. \( \mathcal{Y}(\mathfrak{g}(\mathfrak{t})) \) in its full glory

Answer in \( GW = DT \) theory of local curves in 3-fold.

\[
\text{Target curve } B \quad \text{Mumford} \quad \text{Target 3-fold. } X = \downarrow B
\]

\[
\text{if } L_2 = L_1^V \quad \text{and } t_2 = -t_1 \quad \hbar = -t_1 - t_2
\]
DT theory = enumerative theory of sheaves on 3-folds (and also objects in related categories)

sheaf on $X \cong$ map $B \rightarrow$ moduli of sheaves on $S$

really a map if $\not\subset$ over $B$

otherwise a map with singularities

a monomial ideal sheaf

on $O(-1) \otimes O(-1)$

has degree 2 over $\mathbb{P}^4$
corresponds $\mathbb{P}^1 \longrightarrow \text{Hilb}(\mathbb{C}^2, 2)$
outside of $0, \infty$ $\longrightarrow \begin{array}{c} x_0 \\ x_1 \end{array}$

Boxcounting (especially simple for PT counts)

Both solved explicitly in the language of quantum groups

So far, we discussed Yangians, what about other quantum groups?

$U_q(\hat{sl}(1)) \longrightarrow \text{K-theory of DT Moduli spaces } X = \text{local curve}$

$B \times S^1 = \text{world volume of M2 brane of M-theory}$

The $U_q(\hat{sl}(1))$-theory is the theory of stack of M2-branes on
inside $\mathbb{Z} \times S^1$. CY-5

# of membranes $= L_0$

One of the many motivations to study 2+1 dimensional theories like we do in this course is to better understand M2 branes
\[ \Psi \text{ commutes with quantum group} \]

\[ \rightsquigarrow \text{ a relation in quantum group.} \]