Lecture 18, 09/16

Wednesday, September 16, 2020 10:27 AM

Enumerative geometry & geometric representation theory

Start time Moscow 17:30
New York 10:30

Slices and relations in quantum group

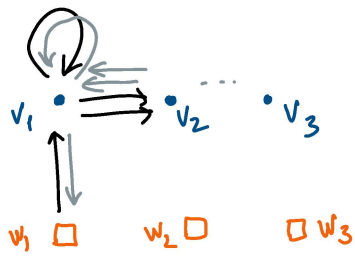
path algebra of Q / relation

quiver data = representation of an algebra

generators = arrow

relations = moment map equations

Q



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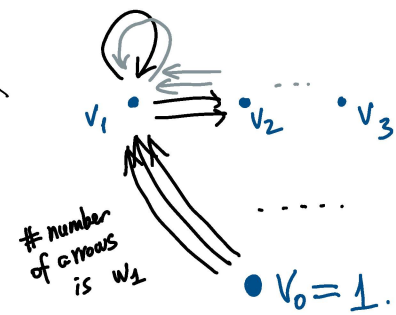


automorphisms

R is a representation of an algebra $\mathcal{A}$, $\quad \mathrm{End}(R) = \mathrm{Hom}_{\mathcal{A}}(R,R)$

1st order deformations $\quad \mathrm{Ext}^1(R,R)$

obstructions to $\longrightarrow \mathrm{Ext}^2(R,R)$

$\vdots$

δ_i representation with $\mathbb{C}$ at ith vertex, all maps are zero

$R = \sum v_i \delta_i \qquad \mathrm{Hom}(R,R) = \oplus\, \mathfrak{gl}(v_i) \supset \prod GL(V_i)$ automorphisms.

Serre duality

$\dim \mathrm{Ext}^1(\delta_i, \delta_j) = \#$ of arrows $i \to j$

observe $\mathrm{Ext}^1(\delta_j, \delta_i) = \hbar^{-1} \cdot \mathrm{Ext}^1(\delta_i, \delta_j)^*$.

obstruction map: moment map to $\mathrm{Ext}^2(R,R) = \hbar^{-1}\,\mathrm{Hom}(R,R)$.

$X =$ Quiver variety $\subset \left[\begin{matrix}\text{quiver data +} \\ \text{moment map eq.}\end{matrix} \Big/ GL(v)\right] =$ stack of representations of $\mathcal{A}$

semisimplification

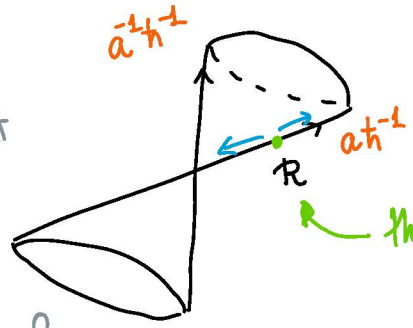
affine quotient $\qquad$ semisimple

singular X_0 = affine quotient i.e. Spec (invariants) = semisimple representations.

(points are closed orbits of $GL(V)$)

$\longleftrightarrow$ 0 $\longleftrightarrow$ $R = \bigoplus_{i=0}^{\cdots} v_i \delta_i$

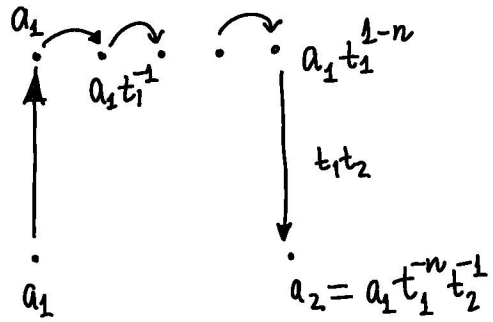
if we specialize equivariant variables then we can have a fixed pt other than 0



the neigborhood of R (and its preimage in X) will look like the quiver variety for the quiver given by $\mathrm{Ext}^1(R_i, R_j)$.

$= \bigoplus v_i' R_i$ irred. new dimension vector

means



$w = 2$

can compute in K-theory because know Ext^0, Ext^2

$$\chi(R, R') = \sum (-1)^i \mathrm{Ext}^i(R, R')$$

$\mathrm{Fock}(a_1) \otimes \mathrm{Fock}(a_2)$ will not be irreducible if

<u>Exersice</u>: Suppose $R = R_0 \oplus \sum_{i \neq 0} v_i' \delta_i$

this has (equivariant) dimension $(1, \beta)$ — 0th vertex

$\Downarrow$ $a_1(1 + t_1^{-1} + \dots + t_1^{1-n})$

Then: quiver is the same up to loops at the 0-th vertex. = vector space coordinates along the stratum containing R

new dim $v' =$

new framing $w' = w - \hbar\, \mathrm{Cartan} \cdot \beta$

$\parallel$

Ext^1

E.g. A_2

$$C = \begin{pmatrix} 1+\hbar^{-1} & -1 \\ -\hbar^{-1} & 1+\hbar^{-1} \end{pmatrix}$$

new family $w' = w - \hbar \, \mathrm{Cartan}$

$$\| $$

$$1 + \hbar^{-1} - \text{adjacency matrix of my quiver}$$

Ext

records $\chi(\delta_i, \delta_j)$ — Hom — Ext^2

A_2 $\quad C = \begin{pmatrix} 1+\hbar^{-1} & -1 \\ -\hbar^{-1} & 1+\hbar^{-1} \end{pmatrix}$

$\hat{A}_0$ $\quad C = 1 + \hbar^{-1} - t_1 - t_2$
$\quad = (1-t_1)(1-t_2)$

For instance: if $\beta = a_1(1 + t_1^{-1} + \dots + t_1^{1-n})$

$$w' = a_1(t_2^{-1} + t_1^{-n})$$

restriction to the neighborhood of a slice

$$\mathrm{Fock}(a) \otimes \mathrm{Fock}(a t_1^{-n} t_2^{-1}) \longrightarrow \mathrm{Fock}(a t_2^{-1}) \otimes \mathrm{Fock}(a t_1^{-n}).$$

order important

a point.

$X(0, w_0)$

$\times$

Stable envelopes are correspondences over X_0

e.g. $$X(\beta, w_0) \times X(v, w) \xrightarrow{\text{Stab}} X(\beta + v, w_0 + w) \xrightarrow{\text{Res}} X(\beta + v, w)$$

take coefficients

contracts to the preimage of 0

$$X_0 \times X_0 \xrightarrow{\text{adds irreducibles}} X_0.$$

$X_0 \ni 0$, $\quad X_0 \ni R \oplus 0$, $\quad R \oplus 0$

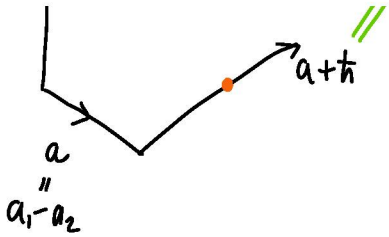
can look at the neighborhood of this point

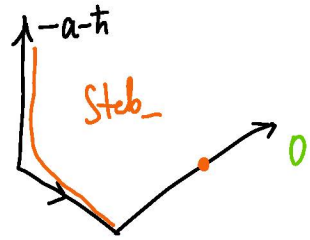
dimensions of 0 add correctly.

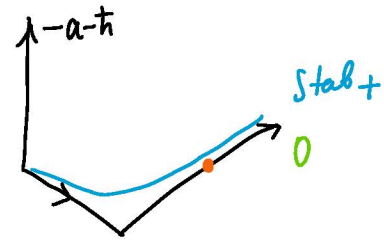
! Restriction to the slice at $R \oplus 0$ is a map of quantum group modules !

<u>Example</u> $\quad Y(\mathfrak{gl}(2)) \hookrightarrow \mathbb{C}^2(a_1) \otimes \mathbb{C}^2(a_2) = H^*(TG(2)^A) \xrightarrow{\text{Stab}} H^*(TG(2)).$

map of Yangian modules in general, an isomorphism.







$$\mathbb{C}^2 \otimes \mathbb{C}^2 \underset{\mathrm{Stab}_+^T}{\overset{\mathrm{Stab}_-}{\rightleftarrows}} H^*(T^*G(2))$$

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \underset{\mathrm{Stab}_+}{\overset{\mathrm{Stab}_-^T}{\rightleftarrows}} H^*(T^*G(2))$$

Stab_- misses the slice, i.e. not surjective.

both maps isomorphisms

Stab_+^T blows up because improper pushforward

Conclusion: $0 \to 3\,\mathrm{dim} \to H^\bullet(T^*G(2)) \xrightarrow{\text{slice}} 1\,\mathrm{dim} \to 0$ always

$0 \to 3 \to \mathbb{C}^2 \otimes \mathbb{C}^2 \to 1 \to 0$ depending on $+/-$

$\updownarrow$ dual

or $0 \to 1 \to \mathbb{C}^2 \otimes \mathbb{C}^2 \to 3 \to 0$

Conjecture: all relations in our quantum groups come from slices [qqqc]

true for $\mathcal{O}$ (enough to check for Y at $\hbar = 0 \Rightarrow$ Plücker)

for $\mathcal{O}$ slices = screening operators

Know by different means that deformations are generically flat

[Nakajima-O] Theorem* if 0 is the only fixed representation then $H^\bullet(\bigsqcup_v X(v,w))$ is irreducible

⋮

Last time: we had a quantum integrable system formed by

$$\mathrm{tr}_1 (z \otimes 1) R_{12}(a) \in \mathrm{End}(\text{2nd factor})(a)$$

$$\mathrm{tr}_1 (z\otimes 1) R_{12}(a) \in \mathrm{End}(\text{2nd factor})(a)$$

commute for all a and fixed z.

fixed operator such that $[z\otimes z, R]=0$

a variable

$\Uparrow$

e.g. z can act by $\prod z_i^{v_i}$

for $z=0$ these are $(\emptyset,\emptyset)$ matrix elements $\Longleftrightarrow$ operators of multiplication in $H^\bullet(X)$, $K(X)$, ... commute

¿ what are these for $z\neq 0$?

Conjectured around 2007/8... by Nekrasov-Shatashvili that these are operators of <u>quantum multiplication</u>

[NS]

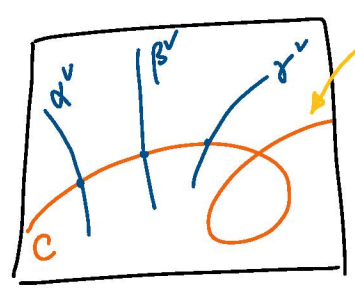
computed spectrum in a "Bethe ansatz form" ($\Leftarrow$ a lot of previous work on diagonalizing Baxter's operators)

(before R-matrices, before our quantum groups,...)

<u>In cohomology:</u> one defines a new associative (!) product on $H^\bullet(X)$ by

$$(\alpha \star \beta, \gamma) = \sum z^{[\deg C]}$$

a formal series in the group algebra of effective cone $\subset H_2(X,\mathbb{Z})$.



rational curve of some degree $\deg C \in H_2(X,\mathbb{Z})$.

$$(\alpha,\beta) = \int_X \alpha\cup\beta$$

degree $C=0$, means $C=pt$

$$= (\alpha\cup\beta,\gamma) + O(z) \ldots$$