Slices and relations in quantum group

Quiver data = representation of an algebra
- Generators = arrow
- Relations = moment map equations

\[ R \text{ is a representation of an algebra } A, \quad \text{End}(R) = \text{Hom}_A(R, R) \]

1st order deformations
- Ext^1(R, R) \quad \text{obstructions to}
- Ext^2(R, R)

\[ \delta_i \text{ representation with } \mathbb{C} \text{ at } i \text{th vertex, all maps are zero} \]

\[ R = \sum v_i \delta_i \quad \text{Hom}(R, R) = \bigoplus \mathfrak{gl}(v_i) = \Pi \text{GL}(v_i) \]

\[ \dim \text{Ext}^4(\delta_i, \delta_j) = \# \text{of arrows } i \to j \]

Observe \[ \text{Ext}^4(\delta_i, \delta_i) = h^{-1} \circ \text{Ext}^4(\delta_i, \delta_j)^\ast \]

Obstruction map: moment map to \[ \text{Ext}^2(R, R) = h^{-1} \text{Hom}(R, R) \]

\[ X = \text{Quiver variety } \subseteq \left[ \text{quiver data + moment map eq. } / \text{GL}(v) \right] = \text{stack of representations of } A \]

\[ \text{nullius quotient } \quad \text{semisimplification} \]
Singular $X_0 = \text{affine quotient} = \text{semisimple representations.}$

\[ \text{Spec (invariants) = points are closed orbits of } \text{GL}(V) \]

\[ R = \bigoplus_{i=0}^{\ldots} v_i \delta_i \]

if we specialize equivariant variables then we can have a fixed pt other than $0$

\[ w = 2 \]

\[ \text{Fock (} a_1 \text{) } \otimes \text{ Fock (} a_2 \text{)} \text{ will not be irreducible if } \]

\[ \text{Exercise: Suppose } R = R_0 \oplus \sum_{i \neq 0} v_i \delta_i \]

\[ \text{Then: quiver is the same up to loops at the 0-th vertex.} \]

new dim $v' = w - \text{Cartan } \beta$

new framing $w' = w - t \text{ Cartan } \beta$

\[ \text{Ext}^1 \]

E.g.

\[ A_2 \rightarrow C = \left( \begin{array}{cc} 1 + t^2 - 1 \\ -t^2 + t^2 \end{array} \right) \]
new primary \( w' = w - h \) ↓

1 + h^{-1} - adjacency matrix of my quiver

\[ A_\circ \rightarrow \quad C = (1 + h^{-1} - 1 h^{-1}) \]
\[ \{ A_\circ : C = 1 + h^{-1} - t_1 - t_2 = (1 - t_1)(1 - t_2) \]

For instance: \( \beta = a_1(t_1 + t_1^2 + \ldots + t_1^n) \)
\( w' = a_1(t_2^{-1} + t_1^n) \)

restriction to the neighborhood of a slice

\[ \text{Fock}(a) \otimes \text{Fock}(a t_2^{-1} t_2^{-1}) \rightarrow \text{Fock}(a t_2^{-1}) \otimes \text{Fock}(a t_2^{-1}) \]

order important

stable envelopes are correspondences over \( X_0 \)

\[ \begin{array}{c}
\text{e.g.} \quad X(\beta, w_0) \times X(\nu, w) \xrightarrow{\text{stab}} X(\beta + \nu, w_0 + w) \xrightarrow{\text{Res}} X(\beta + \nu, w)
\end{array} \]

add irreducibles

take coefficients
contraction to the preimage of \( D \)

one can look at the neighborhood of this point

dimensions of \( D \) add correctly.

restriction to the slice as \( R \otimes 0 \) is a map of quantum group modules!

example \( Y(\mathfrak{sl}(2)) \subset C^2(a_1) \otimes C^2(a_2) = H^* (TG(2)^A) \xrightarrow{\text{stab}} H^* (TG(2)) \)

map of Yangian modules in general, an isomorphism.
\[ C^2 \otimes C^2 \xrightarrow{\text{Stab}_-^T} H^*(T^g(2)) \]

\[ C^2 \otimes C^2 \xrightarrow{\text{Stab}_+} H^*(T^g(2)) \]

\text{Stab}_- \text{ misses the slice, i.e. not surjective.}

\text{Stab}_+^T \text{ blows up because improper push-forward}

\text{both maps isomorphisms}

\text{Conclusion:}
\[ 0 \to 3 \to H^*(T^g(2)) \to 1 \text{ dim} \to 0 \text{ always} \]
\[ 0 \to 3 \to C^2 \otimes C^2 \to 1 \to 0 \text{ depending on } \pm - \]
\[ \text{or} \quad 0 \to 1 \to C^2 \otimes C^2 \to 3 \to 0 \]

\text{Conjecture: all relations in our quantum groups come from slices [999c]}

\text{true for } O \quad (\text{enough to check for } Y \text{ at } \hbar = 0 \Rightarrow \text{Thicker})

\text{for } O \text{ slices = screening operators}

\text{Know by different means that deformations are generically flat}

[Nozumano-0] \text{Theorem: if } 0 \text{ is the only fixed representation then } H^*(U \otimes X(v,w)) \text{ is irreducible}

\text{Last time: we had a quantum integrable system formed by}
\[ \text{tr}_1 (\tau \otimes 1) R_{12}(a) \in \text{End}_2 \text{ (2nd factor)}(a) \]
\[ \text{commute for all } a \text{ and fixed } z. \]

\[ \text{fixed operator such that } [z \otimes z, R] = 0 \]

\[ e.g. z \text{ can act by } T_1 z_i \]

\[ \text{for } z=0 \text{ these are } (\phi, \phi) \text{ matrix elements} \]

\[ \iff \text{operators of multiplication in } H'(X), K(X), \ldots \]

\[ \text{operators of multiplication} \]

\[ \text{commute} \]

Conjectured around 2007/8 ... by Nekrasov- Shatashvili

\[ \text{(before } R\text{-matrices, before our quantum groups, ...)} \]

\[ \text{In cohomology: one defines a new associative } (!) \text{ product on } H^*(X) \text{ by} \]

\[ (\alpha \star \beta, \sigma) = \sum \left[ \deg C \right] \]

\[ \text{a formal series in the group algebra of effective curve } C \in H_2(X, \mathbb{Z}). \]

\[ \text{degree } C = 0, \text{ means } C = pt \]

\[ = (\alpha \cup \beta, \sigma) + O(z) \ldots \]