Lecture 17
Tuesday, September 8, 2020  10:16 AM

\[ t_1 = -t_1 - t_2 \]
\[ t_2 = \frac{1}{t_1} \]
\[ t_3 = \frac{1}{t_1} - t_1 \]

**quiver** ⇒ elliptic/trig/rational \( R(u) \) 𝒄 𝒐 𝒆 𝒇 𝒄 𝒕 𝒐 𝒙 𝒓 𝒑 𝒕 𝒑 𝒑 𝒕 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 𝒑 COVID-19
makes formulas of the following kind smarter

\[ \alpha(t) = \sum_{n} \alpha_n t^{-n} \]

\[ \int :\alpha^2 : , \int :\alpha^3 : , \ldots \]

normal order means put annihilation first

coeff. of \( t^0 \)

\[ -\frac{1}{2} \int :\alpha^2 : = \frac{1}{2} \alpha_0^2 + \sum_{n > 0} \alpha_{-n} \alpha_n \]

i.e.

\( r = \text{classical } R \)-matrix = \( \frac{1}{2} \int :\alpha^2 : \)

while \( R \)-matrix commutes with \( \alpha_+ \)

\( \alpha_\pm = \alpha \otimes 1 \pm 1 \otimes \alpha \)

\( \subset \text{ Fock } \otimes \text{ Fock } \)

TFAE:

\[ \left[ \frac{1}{u^2} \right] R(u) \quad \text{or} \quad \left[ \frac{1}{u^2} \right] R(u) \rho_{\pi, -} \]

or operator of \( c_1(V) \)

This is one operator in a certain quantum integrable system

All operators of cup product

Baxter: suppose \( [z \otimes z, R] = 0 \), then \( \text{tr}_1 (z \otimes 1) R_{12}(u) \in \text{End}(2 \text{nd factor}) \)

commute for fixed \( z \) and all \( u \)

TFAE:

1. \( \ln R(u) = \frac{\hbar}{2u} \int :\alpha^2 : + \frac{\hbar}{6u^2} \int :\alpha^3 : + O \left( \frac{1}{u^3} \right) \)

2. The operator \( c_1(\text{Taut}) = (2 \text{nd quantized}) \text{ Hamiltonian of the quantum trigonometric } \text{CS} \)
The operator $\Delta c_1$:

$$\Delta c_1 = c_4 \otimes 1 + 1 \otimes c_4 - \frac{1}{2} \sum_{n>0} n \alpha_n \otimes \alpha_n$$

5. $\alpha_n \longrightarrow \sum x_i^n$

Many different proofs: e.g. 4 and 5 can be seen directly.

Also 1 \[= R_{gl(c)}^{\hat{c}} = \prod R_{gl(c)} \]

Take $\frac{1}{t^2}$ coeff [Smirnov]

Full R-matrix from YB as follows
Fock $\otimes$ Fock $\xrightarrow{\text{Stab}_Z} H^*_\text{rank 2 instantons}$

\[
\begin{align*}
(R_0 R_1 R_2) & \phi, - , - \\
(R_0 R_1 R_2) & \phi, - , - \\
R_{12} & \end{align*}
\]

or

\[
\begin{align*}
(R_0 R_1 R_2) & \phi, - , - \\
(R_0 R_1 R_2) & \phi, - , - \\
\end{align*}
\]

\[c_1(\text{Taut})\]

explicit operators in Fock $\otimes$ Fock

is

\[
\text{Fock}^+ \otimes \text{Fock}^-
\]

\[\sum \alpha_n^+ \otimes L_{-n}, \leq\]

\[R_{12} \text{ only acts here}\]

**NB.** Sugawara-type Virasoro for $\hat{osp}(1|2)$ is

\[\sum L_n t^{-n} = : \alpha(t);^2\]

\[L_n = \frac{1}{2} \sum_{k \in \mathbb{Z}} \alpha_k \alpha_{n-k} + n \alpha \alpha_n\]

\[L_0 = \frac{1}{2} \alpha^2 + \sum_{k > 0} \alpha_k \alpha_{-k} - \frac{\alpha^2}{2}\]

\[\Delta = \frac{1}{2} \alpha^2 - \frac{1}{4} \alpha^2 \quad c = 1 - 12 \alpha^2\]

\[R \text{ changes } + \rightarrow -\]

this is Virasoro module for Virasoro with

\[\text{lowest weight } \Delta = \frac{1}{2} (u_1 - u_2)^2 - \frac{\alpha^2}{2}\]

central charge

\[1 - 12 \alpha^2\]

framing variables

\[u = u_1 - u_2\]

\[\text{eigenvalue of } \Delta \text{ in } \text{Fock}^-\]

both invariant w.r.t.

\[\alpha \rightarrow -\alpha\]

also

\[u_1 \leftrightarrow u_2\]

\[R \text{- matrix}\]

does nothing in Fock$^+$

does reflection in Fock$^-$

\[\text{Liouville CFT}\]

\[\text{center of mass of two bosons}\]

\[\text{relative position}\]

\[\text{isomorphism between rep. and its dual}\]
Let $R = \text{explicit, because } R = \text{Stab}_< \times \text{Stab}_>$

$= \text{det Shapovalov} = \text{Kac formula} \ [\text{Feigin-Fuchs}]

relations in quantum groups
operators that commute
with $R$-matrices

Geometric source: "shos" inclusion
get action of my quantum group here

$X$ contacts $X_0$ to $0 \in X_0$.
if we make other equivariant
Variables depend on $\hbar$

$X'_0$
neighbourhood of $0$
in $X'_0$
some other quiver variety

$Fock(u_1) \otimes Fock(u_2)$ slide $Fock(u_1-t_2) \otimes Fock(u_1-nt_1)$.

irreducible in general
but reducible if $u_2 = u_1-nt_1-t_2$

screening operators forVirasoro

complete set of relations for $Y(\hat{\mathfrak{g}(1)})$
(e.g. because gives the Plücker relations at $\hbar=0$).

$Y(\hat{\mathfrak{g}(1)}) \rightarrow \mathfrak{u}(\mathfrak{g}(0)) \hookrightarrow \text{End}(\text{Fock})$. 