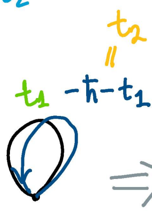
Lecture 17

Tuesday, September 8, 2020 10:16 AM

Enumerative geometry & geometric representation theory

Start time Moscow 17:30
New York 10:30

$\hbar = -t_1 - t_2$



quiver $\Rightarrow$ elliptic / trig / rational $R(u) \hookrightarrow \mathrm{Fock} \otimes \mathrm{Fock}$

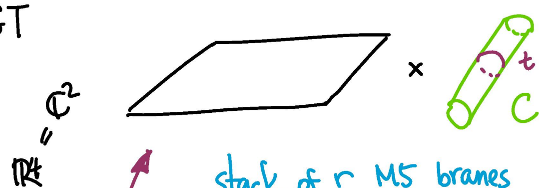
framing depends on $(t_1, t_2) \in T^2 \subset \mathrm{Aut}(\mathbb{C}^2)$ where instantons live.

in cohomology only up to scale

$\mathfrak{X}^2 = \frac{(t_1+t_2)^2}{t_1 t_2}$

goal: understand $R(u)$ in terms of CFT, Virasoro algebra

AGT



C Riemann surface.

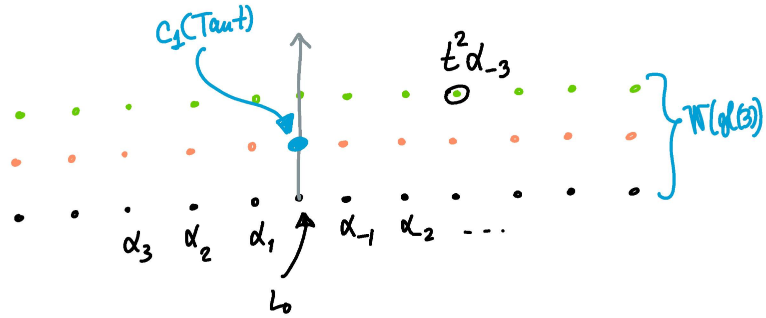
stack of r M5 branes

Nekrasov counts of instantons $=$ CFT on C with $\mathcal{W}(\mathfrak{gl}(r))$-symmetry

Virasoro

$$\begin{array}{ccc} Y(\widehat{\mathfrak{gl}}(1)) & \longrightarrow & \mathrm{End}(\mathrm{Fock}^{\otimes r}) \\ & \searrow \quad \nearrow & \\ & \overline{\mathcal{W}(\mathfrak{gl}(r))} & \end{array}$$

$\overline{\mathcal{W}(\mathfrak{gl}(r))}$ = the image of Y



Today: up to powers of $-1, 2$, and up to cohomology labels

α_{-k}

$\mathrm{Hilb}(n) \times \mathbb{C}^2 \longrightarrow \mathrm{Hilb}(n+k)$

$\widehat{\mathfrak{gl}}(1) = \langle \alpha_n, c, t\frac{d}{dt} \rangle$ $[\alpha_n, \alpha_m] = n\,\delta_{n+m}\, c$

we don't dot α_0 from $[\frac{1}{u}]R$

rank $= r$
$\mathrm{rk}(W)$

$\mathrm{rank}(V) \approx L_0$

$\alpha_0 = c_1(W)$

$\mathrm{Fock}^{\otimes r} \circlearrowleft \alpha_0$ by $a_1 + a_2 + \dots + a_r$ sum of framing variables

$\Delta\alpha_0 = \alpha_0 \otimes 1 + 1 \otimes \alpha_0$, etc.

makes formulas of the following kind smarter $\alpha(t) = \sum_n \alpha_n t^{-n}$

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$\int :\alpha^2:, \quad \int :\alpha^3:, \ldots$ $\qquad$ normal order means put annihilation first

coeff. of t^0

$$\frac{1}{2}\int :\alpha^2: = \frac{1}{2}\alpha_0^2 + \sum_{n>0} \alpha_{-n}\alpha_n$$

i.e. $\quad r = \text{classical R-matrix} = \frac{1}{2}\int :\alpha_-^2: \qquad \alpha_\pm = \alpha\otimes 1 \pm 1\otimes\alpha$

$\circlearrowleft$ Fock $\otimes$ Fock.

whole R-matrix commutes with α_+

<u>TFAE</u> : $\left[\frac{1}{u^2}\right] R(u)$ or $\left[\frac{1}{u^2}\right] R(u)^{\emptyset,-}_{\emptyset,-}$ or operator of $c_1(T)\cup$

Taut

because commutes with α_+

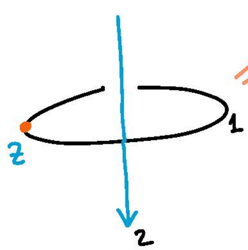
limit case of a Baxter subalgebra

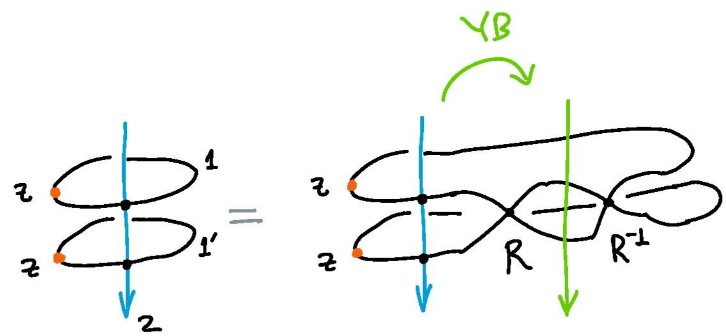
(!) This is one operator in a certain quantum integrable system

all operators of cup product

<u>Baxter</u>: suppose $[z\otimes z, R] = 0$, then $\operatorname{tr}_1 (z\otimes 1) R_{12}(u) \in \operatorname{End}(\text{2nd factor})$.

Commute for <u>fixed</u> z and all u





<u>TFAE</u>: ① $\ln R(u) = \frac{\hbar}{2u}\int :\alpha_-^2: + \frac{\hbar}{6u^2}\int :\alpha_-^3: + O\left(\frac{1}{u^3}\right)$

② the operator $c_1(\text{Taut})$ = (2nd quantized) Hamiltonian of the quantum trigonometric CS

In K-theory

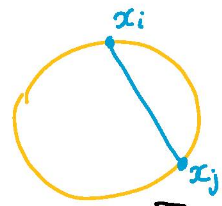
(2) The operator $c_1(\dots)$ — the quantum trigonometric CS

In K-theory Macdonald operators

$$H_{CS} = \frac{1}{2}\int :\alpha^3: + \ae \sum_{n>0} n\,\alpha_{-n}\alpha_n$$

constant

$\approx$ Benjamin-Ono



$U = \text{const}(\ae) \sum \frac{1}{|x_i - x_j|^2}$

eigenfunctions of the form

$\prod_{i<j} (x_i - x_j)^{\text{const}}$ symmetric (x_i)

Jack symmetric poly

H_{CS} $\quad \alpha_{-n} = \sum_{i=1}^{\infty} x_i^n$

(3) $[c_1(\text{Taut}), \alpha_n] \underset{\text{Lehn}}{=}$ Virasoro $\quad [L_n, \alpha_m] \approx -m\,\alpha_{n+m}$

(4) $\Delta c_1 = c_1 \otimes 1 + 1 \otimes c_1 - \hbar \sum_{n>0} n\,\alpha_n \otimes \alpha_{-n}$

compute $c_1 \cup$ in the stable basis of $\text{Fock}^{\otimes 2}$

general, let D be an operator of $\cup$ by a divisor

$$\Delta D = D \otimes 1 + 1 \otimes D - \hbar \sum_{\beta>0} (D,\beta)\, e_\beta \otimes e_{-\beta}$$

$\beta > 0$ effective curves

canonical tensor in $\mathfrak{g}_\beta \otimes \mathfrak{g}_{-\beta}$

(5) $\alpha_{-n} \longrightarrow \sum x_i^n$

fixed pts $\longrightarrow$ Jack polynomials (determined by triangularity and orthogonality)

$\shortparallel$

eigenfunctions of $\cup$

Many different proofs: e.g. (4) and (5) can be seen directly

also (1) $\Longleftarrow$ $R_{\widehat{\mathfrak{gl}(1)}} = \overrightarrow{\prod} R_{\mathfrak{gl}(\infty)}$ $\quad \bigcirc = \text{—•—•—}/\mathbb{Z}$

Take $\frac{1}{u^2}$ coeff [Smirnov]

Full R-matrix from YB as follows

$$\mathrm{Fock}\otimes\mathrm{Fock} \xrightarrow{\mathrm{Stab}_{\gtrless}} H^{\bullet}\left(\begin{matrix}\text{rank } 2\\ \text{instantons}\end{matrix}\right)$$

$$\left(R_{01}R_{02}\right)^{\varnothing,-,-}_{\varnothing,-,-}$$

R_{12}

or

$$\left(R_{02}R_{01}\right)^{\varnothing,-,-}_{\varnothing,-,-}$$

$c_1(T_{\mathrm{ant}})$

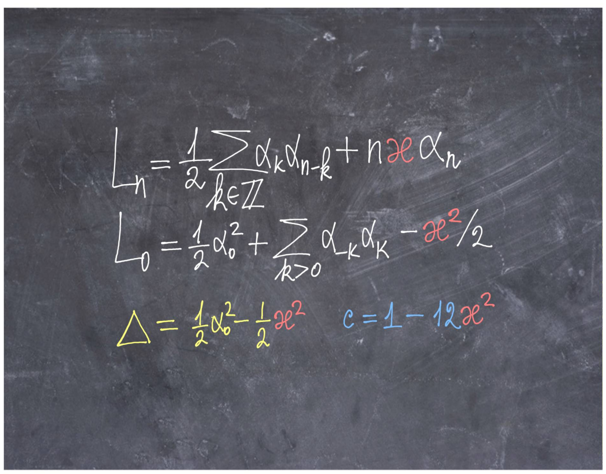
explicit operators in Fock $\otimes$ Fock

$\text{Fock}^+ \otimes \text{Fock}^-$

R_{12} only acts here

$$\dots + \sum \alpha^+_n \otimes L^-_{-n,\lessgtr}$$

NB. Sugawara-type Virasoro for $\widehat{\mathfrak{gl}(1)}$ is $\sum L^\circ_n t^{-n} = :\alpha(t):^2$



$$L_{n,\gtrless} = \frac{1}{2}\sum \alpha_k\alpha_{n-k} \pm n\varkappa\alpha_n$$

R changes + to −

$\text{Fock}^+ \otimes \text{Fock}^-$ this is Verma module for Virasoro with

Lowest weight $\Delta = \frac{1}{2}(u_1-u_2)^2 - \frac{\varkappa^2}{2}$

central charge $1 - 12\varkappa^2$

both invariant w.r.t $\varkappa \to -\varkappa$

also $u_1 \leftrightarrow u_2$

$\mathrm{Fock}(u_1)\otimes\mathrm{Fock}(u_2)$

framing variables $u = u_1 - u_2$

eigenvalue of α_0^- in Fock^-

R-matrix does nothing in Fock^+ — center of mass of two bosons

does reflection in Fock^- — relative position

Liouville CFT

isomorphism between repr and its dual

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$$\det R = \text{explicit, because } R = \mathrm{Stab}_<^{-1} \circ \mathrm{Stab}_>$$

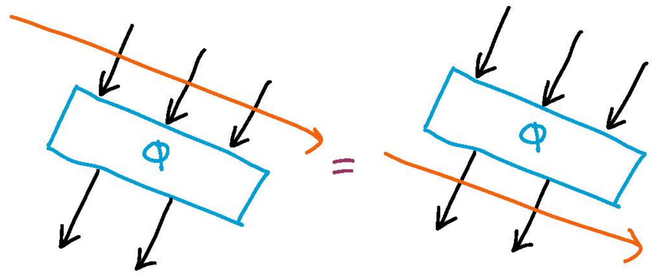
triangular

$$= \det \text{Shapovalov} = \text{Kac formula} \quad \text{[Feigin-Fuchs]}$$

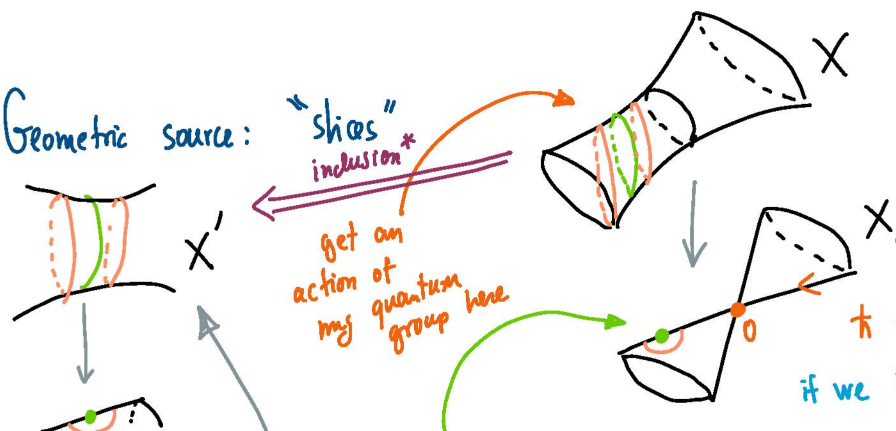
relations in quantum groups

$\Updownarrow$

operators that commute with R-matrices



Geometric source: "slices"



inclusion*

get an action of my quantum group here

neighborhood of 0 in X_0'

$\hbar$ contracts X_0 to $0 \in X_0$.

if we make other equivariant variables depend on $\hbar$

fixed pts of specialised variables

Some other quiver variety

$$\mathrm{Fock}(u_1) \otimes \mathrm{Fock}(u_2) \xrightarrow[\text{slice}]{} \mathrm{Fock}(u_1 - t_2) \otimes \mathrm{Fock}(u_1 - nt_1).$$

irreducible in general

but reducible if $u_2 = u_1 - nt_1 - t_2$

screening operators for Virasoro

$\Downarrow$

complete set of relations for $Y(\widehat{\mathfrak{gl}}(1))$

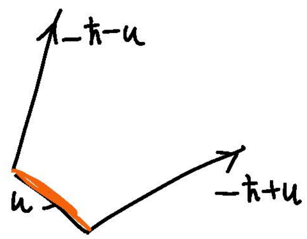
(e.g. because gives the Plücker relations at $\hbar = 0$).

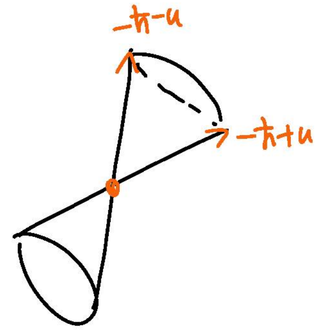
$$Y(\widehat{\mathfrak{gl}}(1)) \longrightarrow \mathcal{U}(\mathfrak{gl}(\infty)) \hookrightarrow \mathrm{End}(\mathrm{Fock}).$$

$$\mathrm{Fock} = \Lambda^{\frac{\infty}{2}} \mathbb{C}^\infty$$

$$Y(\widehat{gl}(1)) \xrightarrow[\hbar=0]{} \mathcal{U}(gl(\infty)) \hookrightarrow \mathrm{End}(\mathrm{Fock}) = \Lambda^{\frac{\infty}{2}}\mathbb{C}^{\infty}$$

$$\mathrm{End}(M) \hookrightarrow \mathrm{End}(\Lambda^{\bullet} M)$$





$\mathbb{C}^2(u_1)\otimes\mathbb{C}^2(u_2)$

something happens $u_1-u_2=\pm\hbar$

$0\to 3 \to H^{\bullet}(TG(2)) \to 1 \to 1$

$$H^{\bullet}(M(1))^{\otimes 2} \underset{Stab_<}{\overset{Stab_>}{\rightrightarrows}} H^{\bullet}(M(2))$$