Lecture 16
Wednesday, September 2, 2020

Nakajima Variety

\[ X \rightarrow \text{projective, also} \]

\[ \mathbb{P}^1 \]

\[ X_0 \]

\[ \text{Spec } R_0 \]

\[ \text{affine} \]

\[ \text{invariants of } G \]

\[ \\]

\[ \text{GIT quotient, i.e.} \]

\[ \text{Proj} \otimes R_k \]

\[ T^+ \mathbb{P}^1 \]

\[ uv = w^2 \]

Axiomatization: \( X \) is called an equivariant symplectic resolution if

1. \( X \) is algebraic symplectic (e.g. a smooth alg. sympl. reduction)
2. \( X \to X_0 = \text{Spec } H^0(O_X) \) is proper and birational
3. \( \exists \sigma(\xi) \in \text{Aut}(X) \) that contracts \( X_0 \) to a point

\[ \Rightarrow \text{D. Kaledin, "Geometry and topology of symplectic resolutions"} \]

Thm. \( \text{Steinberg}(X) \overset{\text{def}}{=} X \times X \subset X \times X \) is isotropic w. r. t. \((\omega_X, -\omega_X)\)

\[ \{(x_1, x_2), p(x_1) = p(x_2)\}. \Rightarrow \text{dim } X \leq \text{dim } X \]

Now suppose \( L \subset X \times Y \) is a correspondence (may be singular,...)

it can be 1. Lagrangian
2. Steinberg, i.e. \( L \subset X \times Y \) where \( X \to Y \to Y \) proper

is the prototypical example in \( H^2_{eq} \). !!!!
is the prototypical example

Lagrangian Steinberg correspondences commute with $R$-matrices in the following sense:

**Def.** Let $X$ be symplectic, $A \subset X$ preserving $\omega$, $L \subset X$ Lagrangian $A$-invariant.

\[ L^A = U L_i \]

Isotropic, top dimensional Lagrangian

\[ \text{Res} L = \sum m_i L_i \]

Multiplicity.

Is there a smooth point on $L_i$?

Vector space polynomials in $x = \text{Lie} A$

**Lemma:** $[F_i] = m_i [N_{f_i < 0}]$

Better $[T_{f_i}^{1/2} f_i \neq 0]$

**Thin [989c]**

\[ X \to Y \]

\[ \text{Stab}_{C, T_x^{1/2}} \to \text{Stab}_{-C, T_{-y}^{1/2}} \]

Chain of attracting directions

Commuting for all $C$!

Corollary,

\[ \text{Res} L R_X = R_Y \text{Res} L \]

Rigidity

Use Steinberg for properness

Send $a \to \infty$.

Torsion-free and framed at $l_\infty = p^2 / c^2$

$GL(r)$ acts on $U_A$ maximal torus

...
Example:

\[(v_k, w_k) = (n, r)\]

\[X(v_k, w_k) = M(r, n) = \text{moduli space of certain sheaves } \mathcal{F} \text{ on } \mathbb{C}^2 \text{ or } \mathbb{P}^2 \text{ of rank } 2 \]

\[\text{and } c_2(\mathcal{F}) = n\]

\[\text{i.e. } M(1, n) = \{ I \subseteq \mathbb{C}[x_1, x_2] \} = \text{Hull}(\mathbb{C}^2, n)\]

\[\text{ideals } c_2 = \text{codim} = n\]

\[M(r) = \bigcup_n M(r, n)\]

\[M(r) = M(1) \times M(1) \times \ldots \times M(1) \quad \text{for } r \text{ times}\]

\[\mathcal{F} = \bigoplus I_i\]

\[M(r, n) \times \mathbb{C}^2 \overset{\alpha_{-k}}{\longrightarrow} M(r, n+k)\]

\[\mathcal{F}_1 \quad \mathcal{F}_2\]

\[0 \to \mathcal{F}_2 \to \mathcal{F}_2 \to \text{torsion sheaf supported at } p \to 0\]

\[M(r+1, n+k)\]

\[\text{for } r = 1, \text{ Barannnikov, } \ldots, \text{ for } r > 1\]

\[\Rightarrow \text{isotropic, Steinberg, middle dimensional} \Rightarrow \text{LS correspondence}\]

\[M(2) = M(1) \times M(1)\]

\[\text{Res } \alpha_{-k} = \ldots \alpha_{-k} \otimes 1 + \ldots 1 \otimes \alpha_{-k}\]

\[\text{the coeff is } \pm 1 \text{ because there is a smooth point}\]

\[= \alpha_{-k} \otimes 1 + 1 \otimes \alpha_{-k}\]

So these operators commute with \( R^{-1} \)
So these operators commute with $R^{-1}$ matrices

$$
\alpha_k = \pm \alpha_{-k}^{\top}
$$

$k > 0$

$$
[\alpha_k, \alpha_l] = n \cdot \delta_{k+l} \cdot \text{diag} \, \hat{\alpha}^2
\Rightarrow \text{rank}
$$

$$
H^*_\text{eq.}(M(1)) = \bigoplus H^*_\text{eq.}(\text{hilb}(C^2, \text{n}))
\Rightarrow \text{Fock module for these operators in particular irreducible}
\Rightarrow \text{Fock}
\Rightarrow M(1) \otimes M(1)
$$

$R \subset M(1) \otimes M(1)$ commutes with $\alpha_n^+$, $\alpha_n^-$,

$$
\alpha_n^+ = \frac{\alpha_n \otimes 1 \pm 1 \otimes \alpha_n}{\sqrt{2}}
$$

$\text{Fock \otimes Fock} = \text{Fock}^+ \otimes \text{Fock}^-$

$$
[\alpha_n^+, \alpha_n^-] = 0
\Rightarrow R = \text{some expression in } \alpha_n^-
$$

Lemma: any expression in $\alpha_n^-$ is uniquely determined by its matrix elements in vac $\otimes$ Fock

$$
R(u) = 1 - \frac{12}{u}
$$

for $\gamma(\text{dim}(n))$

$$
\frac{n_i + u + k}{w_i + u_i} = 1 + \frac{k}{u} \cdot \alpha_n + \frac{k}{u^2} \left( \star c_1 + \star r_2 \right) + \ldots
\Rightarrow R(u) = 1 - \frac{1}{u} \sum \alpha_n \alpha_n^* \downarrow
$$
\[
\frac{\prod_{i=1}^k \left( w_i + u + h \right)}{w_i + u} = 1 + \frac{h}{u} r_k + \frac{h}{u^2} (\star c_1 + \star r_k) + \ldots
\]

\[
R(u) = 1 - \frac{t}{u} \sum_{n \geq 0} a_n^+ a_n^-
\]

up to normalization

One step remaining to the full \( R \)-matrix!

\[
YB \quad R_{12}(u_1-u_3) R_{13}(u_1-u_3) R_{23}(u_2-u_3) = R_{23}(u_2-u_3) \ldots \]

\[
u_1 \to \infty \quad u = \infty \quad \checkmark
\]

coef of \( \frac{1}{u_1} \)

\[
[ r_{12} + r_{13}, R_{23} ] = 0
\]

coef of \( \frac{1}{u_1^2} \)

\[
(\ldots) R_{23} = R_{23} (\ldots)
\]

is a certain Virasoro intertwiner

Virasoro acting

has to do with mult by \( c_1(T_{\text{int}}) \) on \( M(2) \)

Fock \( \otimes \) Fock

irreducible

\[
M(v_1, w_1) \times M(v_2, w_2) \quad \Theta
\]

\[
M(v_1 + v_2, w_1 + w_2) \leftarrow M(v_1', w_1') \times M(v_2', w_2')
\]

Steinberg

\[
M(\alpha, w_0) \times M(v, w) \quad \text{Act}
\]

\[
M(0, w_0) \times M(\alpha + v, w)
\]

upper/lower diagonal part of \( r \) is supported on a LS correspondence.
\[
M(\mathfrak{g}, \mathcal{W}_0) \times M(\mathfrak{g}, \mathcal{W}_0) \xrightarrow{P} M(\mathfrak{g}, \mathcal{W}_0) \times M(\mathfrak{g}, \mathcal{W}_0)
\]

\[P^2 = -\mathcal{W}_0(\mathfrak{g})P\]

\[P\] projects onto \[\mathcal{O}_\mathfrak{g} \subset \mathcal{H}^0_{\text{eq}}(M(\mathfrak{g}, \mathcal{W}_0)).\]

Both \(P\) and \(\mathcal{O}_\mathfrak{g}\) are residues of the correspondence "two points lie on a rational curve of degree \(\alpha\)" virtual cycle.

For \(Q\) can be eyeballed directly \(\Rightarrow\) reproduces the formula for \(P\)

\[P = \alpha_n \alpha_m\]

Rational \(R\)-matrix for \(Q\)

\[\mathcal{Y}(\mathcal{O}_\mathfrak{g}(1)) = \text{deformation } U(\mathcal{Y}[t])\]

Virasoro

\[W(\mathcal{O}_\mathfrak{g})\]

\[W(\mathcal{O}_\mathfrak{g}r)\]

This gives the action of \(\mathcal{O}_\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g} \oplus \cdots\)

\([\text{coeff } u] \mathcal{O}_\mathfrak{g}\)

\([\text{coeff } u^2] \mathcal{O}_\mathfrak{g}\)