Lecture 16

Wednesday, September 2, 2020 10:25 AM

Enumerative geometry & geometric representation theory

Start time Moscow 17:30
New York 10:30

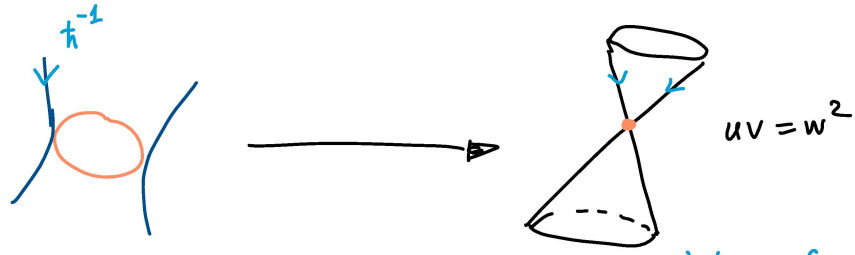
Nakajima variety

$$X \xrightarrow[\text{birational onto its image}]{\text{projective, also}} X_0$$

$X \to \cdots /\!/ G$ — GIT quotient, i.e. $\mathrm{Proj} \bigoplus_{k \geq 0} R_k$

X_0: invariants of G; $\mathrm{Spec}\, R_0$ affine

$T^*\mathbb{P}^1$



Axiomatization: X is called an equivariant symplectic resolution if

1. X is algebraic symplectic (e.g. a smooth alg. sympl. reduction)
2. $X \xrightarrow{\rho} X_0 = \mathrm{Spec}\, H^0(\mathcal{O}_X)$ is proper and birational
3. $\exists\, \sigma(t) \in \mathrm{Aut}(X)$ that contracts X_0 to a point

$\Rightarrow$ D. Kaledin, "Geometry and topology of symplectic resolutions"

Thm. $\mathrm{Steinberg}(X) \overset{\text{def}}{=} X \times_{X_0} X \subset X \times X$ is isotropic w.r.t $(\omega_X, -\omega_X)$

$= \{(x_1, x_2),\ \rho(x_1) = \rho(x_2)\}$.

$\Rightarrow \dim \leq \dim X$

top dimensional components are Lagrangian

Now suppose $L \subset X \times Y$ is a correspondence (may be singular, ...)

it can be
1. Lagrangian
2. Steinberg, i.e. $L \subset X \times_V Y$ where $X \xrightarrow{\text{proper}} V$, $Y \xrightarrow{\text{proper}} V$

is the prototypical example

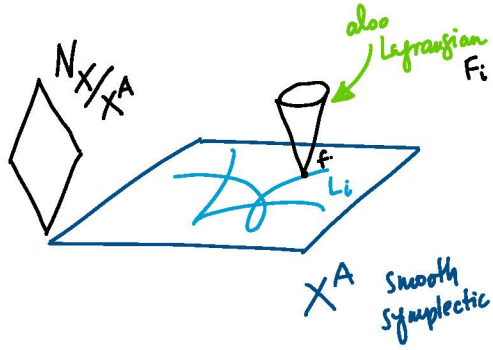
in $H^\bullet_{eq}$!!!

is the prototypical example

in $H^\bullet_{eq}$!!!

Lagrangian Steinberg correspondences commute with R-matrices in the following sense:

Def. Let X be symplectic, $A \circlearrowright X$ preserving ω, $L \subset X$ Lagrangian
torus A-invariant



$$L^A = \bigcup L_i$$

isotropic, top dimensional Lagrangian

$$\mathrm{Res}\, L = \sum m_i L_i$$

multiplicity.

± 1 if there is a smooth point on L_i

$[F_i] \in H^\bullet_A(N_f)$ **Lemma:** $[F_i] = m_i [N_{f,<0}]$ [qgqc]

better $m_i [T^{1/2}_{f,\neq 0}]$

polynomials in $\mathfrak{a} = \mathrm{Lie}\, A$ vector space

Thm [qgqc]

$$\begin{tikzcd}
X^A \arrow[r] \arrow[d, "\mathrm{Res}_{T^{1/2}_{X}, T^{1/2}_{Y,opp}} L"'] & X \arrow[d, "L"] \\
Y^A \arrow[r] & Y
\end{tikzcd}$$

$\mathrm{Stab}_{C, T^{1/2}_X}$ choice of attracting directions

a Lagrangian Steinberg correspondence.
A-equivariant

$\mathrm{Stab}_{C, T^{1/2}_Y} = \mathrm{Stab}^{-T}_{-C, T^{1/2}_{Y_{opp}}}$.

Commutes for all C !

rigidity

Corollary, $\mathrm{Res}\, L\, R_X = R_Y\, \mathrm{Res}\, L$

use Steinberg for properness
send $a \in \mathfrak{a} \to \infty$.

$\mathcal{F}|_{\ell_\infty} \xrightarrow{\varphi} \mathcal{O}^{\oplus r}_{\ell_\infty}$

torsion-free and framed at $\ell_\infty = \mathbb{P}^2 \setminus \mathbb{C}^2$

$GL(r)$ acts on
$\cup$
A maximal torus

contain charges $\mathcal{F}$

$\overset{\cup}{A}$ maximal

Example: 

$(v_1, w_1) = (n, r)$

$X(v_1, w_1) = M(r,n) =$ moduli space of certain sheaves $\mathcal{F}$ on $\mathbb{C}^2$ or $\mathbb{P}^2$ of rank 2 and $c_2(\mathcal{F}) = n$

i.e. $M(1,n) = \{ \underset{\text{ideals}}{I \subset \mathbb{C}[x_1, x_2]} \} = \mathrm{Hilb}(\mathbb{C}^2, n)$. $\quad c_2 = \text{codim} = n$

$M(r) = \bigsqcup_n M(r,n)$

framing torus

$$M(r)^A = \underbrace{M(1) \times M(1) \times \dots \times M(1)}_{r \text{ times}} \qquad \mathcal{F} = \bigoplus_{i=1}^{r} I_i$$

$$\begin{tikzcd}
M(r,n) \times \mathbb{C}^2 \arrow[r, "\alpha_{-k}"] \arrow[dd] & M(r, n+k) \arrow[dd] \\
\mathcal{F}_1 \qquad p & \mathcal{F}_2 \\
& M(r+1, n+k)
\end{tikzcd}$$

is Lagrangian Steinberg correspondence.

$$0 \to \mathcal{F}_2 \to \mathcal{F}_1 \to \text{torsion sheaf supported at } p \to 0$$

Nakajima for $r=1$, Baranovsky, ... $r>1$

$\mathcal{F}_1 \oplus$ some fixed I translated to p $\qquad \mathcal{F}_2 \oplus \mathcal{O}$

$p = (p_1, p_2)$ e.g. $(x_1 - p_1)(x_2 - p_2)^k$ $\qquad (\mathcal{F}^{vv}, \text{same})$

$M(r,n) \longrightarrow M(r,n)_0$ Uhlenbeck space

$\mathcal{F} \longrightarrow (\mathcal{F}^{vv}, \text{supp } \mathcal{F}^{vv}/\mathcal{F})$

vector bundle $\qquad$ torsion

$\Rightarrow$ isotropic, Steinberg, + middle dimensional $\Longrightarrow$ LS correspondence

$M(2)^A = M(1) \times M(1)$

$\mathrm{Res}\, \alpha_{-k} = \dots \alpha_{-k} \otimes 1 + \dots 1 \otimes \alpha_{-k}$

the coeff is ± 1 because there is a smooth point

So these operators commute with R-

$= \alpha_{-k} \otimes 1 + 1 \otimes \alpha_{-k}$

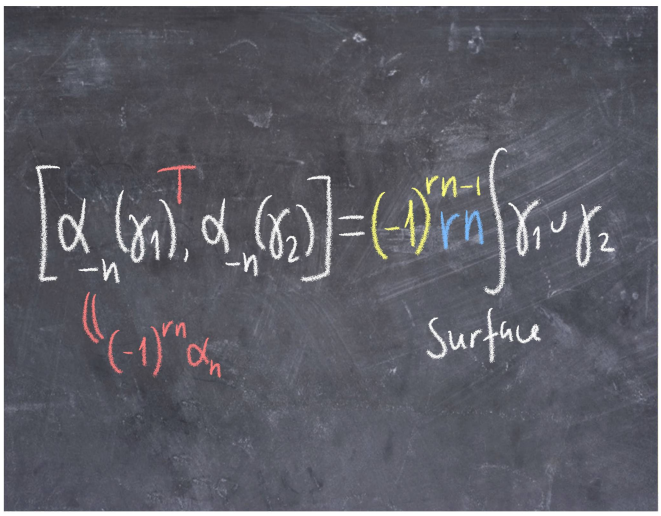
So these operators commute with R-matrices !

$= \alpha_{-k} \otimes 1 + 1 \otimes \alpha_{-k}$

define $\alpha_k = \pm \alpha_{-k}^T$
$k > 0$

$[\alpha_k, \alpha_\ell] = r\, k\, \delta_{k+\ell}\, \mathrm{diag}_{\mathbb{C}^2}$

$\mathbb{C}^2$ $\mathbb{C}^2$ rank



$H^\bullet_{eq}(M(1)) = \bigoplus_n H^\bullet_{eq}(\mathrm{Hilb}(\mathbb{C}^2, n))$ is a Fock module for these operators in particular irreducible

$\parallel$
Fock

$R \hookrightarrow M(1) \otimes M(1)$ commutes with α_n^+ , $\alpha_n^\pm = \dfrac{\alpha_n \otimes 1 \pm 1 \otimes \alpha_n}{\sqrt{2}}$

center of mass $\varphi^{(1)} - \varphi^{(2)}$

not really needed

$\mathrm{Fock} \otimes \mathrm{Fock} = \mathrm{Fock}^+ \otimes \mathrm{Fock}^-$

$\varphi^{(1)}$ $\varphi^{(2)}$ α_n^+ α_n^-

$[\alpha_n^+, \alpha_\ell^-] = 0$

irreducible

R = some expression in α_n^-

$$\boxed{R(u) = 1 - \frac{(12)}{u} \quad \text{for } Y(\mathfrak{gl}(n))}$$

<u>Lemma:</u> any expression in α_n^- is uniquely determined by its matrix elements in $\mathrm{vac} \otimes \mathrm{Fock}$

$\downarrow$

$\mathrm{vac} \otimes \mathrm{Fock}$.

$R^{\emptyset,\cdot}_{\emptyset,\cdot}$

operators of cap product by characteristic class of Taut.

$\alpha_{-n}^-(pt)\, \alpha_n^-(1)$.

$$\prod^{rk} \frac{w_i + u + \hbar}{w_i + u} = 1 + \frac{\hbar}{u} rk + \frac{\hbar}{u^2}(* c_1 + * rk) + \dots$$

$$R(u) = 1 - \frac{\hbar}{u} \sum \alpha_{-n}^- \alpha_n^-$$

$$\prod_{i=1}^{rk} \frac{w_i+u+\hbar}{w_i+u} = 1 + \frac{\hbar}{u} rk + \frac{\hbar}{u^2}\left(* c_1 + * rk\right) + \dots$$

$$rk\, Taut = n = \sum_{k>0} \alpha_{-k}\alpha_k = L_0$$

up to normalization

$$R(u) = 1 - \frac{\hbar}{u} \sum_{n>0} \alpha^-_{-n}\alpha^-_n$$

$\mathrm{diag}_{\mathbb{C}^2}$

L_0^-

One step remaining to the full R-matrix !

YB $\quad R_{12}(u_1-u_2)\, R_{13}(u_1-u_3)\, R_{23}(u_2-u_3) = R_{23}(u_2-u_3) \dots$

$u_1 \to \infty \qquad u=\infty \qquad \checkmark$

coeff of $\frac{1}{u_1}$ $\quad [r_{12}+r_{13}, R_{23}] = 0$

R commutes with Nakajima - Baranovsky oper.

coeff of $\frac{1}{u_1^2}$ $\quad (\dots)R_{23} = R_{23}(\dots)$

not the same

has to do with mult by $c_1(Taut)$ on $M(2)$

$Fock^{\otimes 2} = Fock^+ \otimes Fock^-$

Virasoro acting

irreducible.

is a certain Virasoro intertwiner

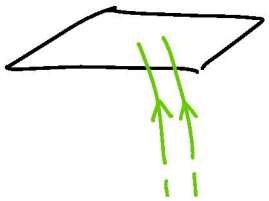
$$M(v_1,w_1) \times M(v_2,w_2) \xrightarrow{\oplus} M(v_1+v_2, w_1+w_2) \leftarrow M(v_1',w_1') \times M(v_2',w_2')$$

$\cup$

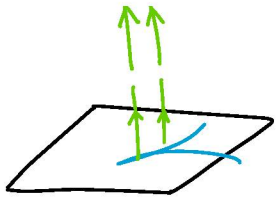
Steinberg

make a act here by a scalar

$$M(\underline{\underline{\alpha}}, w_0) \times M(v,w) \xrightarrow{Act} M(\overset{pt}{0}, w_0) \times M(\alpha+v, w)$$



upper/lower diagonal part of r is supported on a LS correspondence.



"part of r is supported on correspondence."

this gives the action of $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha$ by taking the coeff in $M(\alpha, w_0)$

$$M(\gamma, w_0) \times M(0, w_0) \xrightarrow{P} M(0, w_0) \times M(\gamma, w_0)$$

YB $\Rightarrow$ $\underline{P^2 = -w_0(\alpha) P}$ $\quad$ P projects onto $\mathfrak{g}_\alpha \subset H^\bullet_{eq}(M(\alpha, w_0))$.

Both $\underline{P}$ and $\underline{\mathrm{Act}}$ are residues of the correspondence "two points lie on a rational curve of degree α"

virtual cycle.

for Q can be eyeballed directly $\Rightarrow$ reproduces the formula for r

$P = \alpha_{-n}\alpha_n$

coeff $\frac{1}{u}$

Rational R-matrix for Q $\longrightarrow$ $Y(\widehat{\mathfrak{gl}(1)})$ $= \mathfrak{gl}(1) = \langle \alpha_n \rangle$ $=$ deformation $\mathcal{U}(\mathfrak{g}[t])$

Virasoro

$W(\mathfrak{gl}_2)$

$W(\mathfrak{gl}_r)$

$\vdots$

$t\alpha_n$

coeff $\frac{1}{u^2}$