Lecture 15

Wednesday, August 26, 2020 9:30 AM

Enumerative geometry & geometric representation theory

Start time Moscow 17:30

New York 10:30

$X \approx$ moduli space of vacua $\supset \mathrm{Crit}(\Phi)/U$ compact gauge group

$M =$ " matter" $=$ a symplectic representation of G complex $\supset U$ max compact

$\Longrightarrow$ a very rich world of examples

$X = M /\!\!/\!\!/\!\!/ G = \mu^{-1}(0) /\!\!/ G = \mu_{\mathbb{H}}^{-1}(0 \oplus \theta)/U$

moment map GIT central

$\mu_{\mathbb{H}} = \mu_{\mathbb{C}} \oplus \mu_{\mathbb{R}} \longrightarrow \mathfrak{g}^* \otimes (\mathbb{C} \oplus \mathbb{R})$

$\Phi = \langle \mu_{\mathbb{H}}, \xi_{\mathbb{H}} \rangle, \quad \xi_{\mathbb{H}} \in \mathfrak{g} \otimes (\mathbb{C} \oplus \mathbb{R})$ SUSY partners of gauge fields

$\cup$

X is smooth is rare

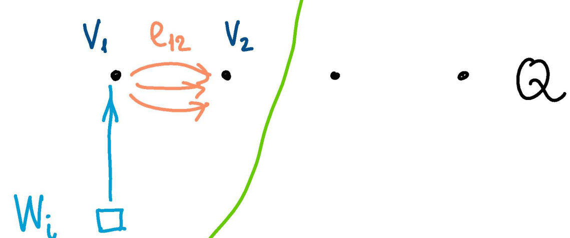
e.g. no stabilisers in the $\mu^{-1}(0) \cap$ semistable $\rightsquigarrow$ M has to be small

$\mathrm{Hom}(W_i, V_i)$ $GL(V_i)$ does not act here

For $G = \prod GL(V_i)$ $\quad M = V_i^{\oplus w_i} \oplus$ dual

$\mathrm{Hom}(V_i, V_j)^{\otimes e_{ij}} \oplus$ dual

$X =$ Nakajima quiver variety



G commutes with $\prod GL(W_i) \times \prod GL(E_{ij})$

fixed points of these elements are quiver varieties with $\tilde{Q} \xrightarrow{\text{cover}} Q$.

fixed points here are $\prod$ quiver varieties of same type

$\oplus W_i$

$/\!\!/$

e.g. $Q = \bullet$ (no arrows)

$\oplus W_i$ $\qquad \prod$ quiver varieties of same type

$\parallel$

e.g. $Q = \bullet$ (no arrows)

$W = W' \oplus a W''$

$$X(w) = \bigoplus_v X(v,w) \qquad X(w) = T^*G(w) = \bigsqcup_v T^* \mathrm{Grass}(v,w)$$

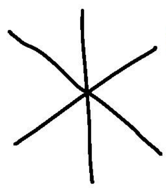
$$\begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & a & & & \\ & & & & a & & \\ & & & & & a & \\ & & & & & & a \end{pmatrix} \quad \text{1's: } w_i', \quad \text{a's: } w_i''$$

$$X(w)^a = X(w') \times X(w'') \qquad \Longleftarrow \quad \text{Exercise}$$

$\equiv$

$R(a)$ $\qquad$ R-matrix satisfying YB eq. etc.

$W = W' \oplus a_2 W'' \oplus a_3 W'''$



$a_2 = a_3 \rightsquigarrow X(w') \times X(w''+w''')$

R_{23}

$\Longrightarrow$ a quantum group $\qquad$ Goal: this quantum group is an elliptic deformation $\mathcal{U}(\hat{\mathfrak{g}})$

$\leftarrow$ loops into $\mathfrak{g}$

$\mathfrak{g}$ = Lie algebra

which has something to do with the quiver (not the KM).

$A_1 \quad A_2$

$\bullet\ , \quad \bullet\!-\!\bullet$

$\mathfrak{g}_{KM} = \mathfrak{sl}(3)$

$\cap$

$\mathfrak{g} = \mathfrak{gl}(3)$

do more or less by hand

$\hat{A}_0\ , \qquad \hat{A}_2\ ,$

$$\mathfrak{g} \approx \widehat{\mathfrak{gl}(1)} = \left\langle \alpha_n, \frac{d}{dt}, c \right\rangle \Big/\!\!\Big/$$

loops in $\mathfrak{gl}(1)$

rotation

central extension

$\Uparrow$

the most important case for enumerative applications

$$[\alpha_n, \alpha_m] = n\delta_{n+m} c$$

$$\left[\frac{d}{dt}, \alpha_n\right] = n\alpha_n$$

$$\mathfrak{g}_{KM} = \left\langle \alpha_{\pm 1}, \frac{d}{dt}, c \right\rangle$$

the difference is much more dramatic for other quivers $\quad \mathfrak{g} \gg \mathfrak{g}_{KM}$

For $\bigcirc$ $\qquad X(1) = \bigsqcup_{n \geq 0} \mathrm{Hilb}(\mathbb{C}^2, n)$ $\qquad$ and $\qquad X(r)$ = moduli space of certain sheaves $\mathcal{F}$ on $\mathbb{C}^2$ of rank r

$\qquad \qquad \qquad \qquad \qquad \qquad \quad \Psi$

ideal $\quad I \subset \mathbb{C}[x_1, x_2]$

of dimension n

ideal $I \subset \mathbb{C}[x_1, x_2]$
of codimension n

sheaves ...
of rank r

Goal: understand the quantum group as a deformation of $\mathcal{U}(\hat{\mathfrak{g}})$

"$\widehat{\mathfrak{gl}(1)}$"

find a place where $R = 1$

Elliptic quantum group $\rightsquigarrow$ $\mathcal{U}_q(\hat{\mathfrak{g}})$ $\rightsquigarrow$ $Y(\mathfrak{g})$ = a graded deformation of $\mathcal{U}(\hat{\mathfrak{g}})$

curve E of $\mathrm{Ell}_T(X)$ — $\mathbb{G}_m$ — $\mathbb{G}_a$ additive formal group
$\Rightarrow$ ordinary equivariant cohom.

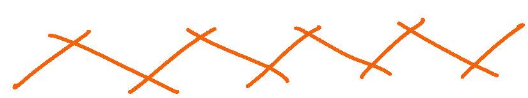
here, stable envelope was
a section of some line
bundle $\mathcal{S}$

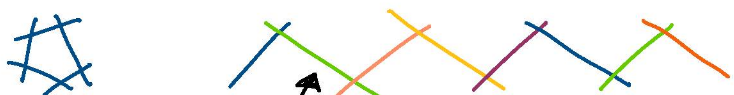
$\mathrm{Stab}|_{\text{fixed component } F_i}$ $\rightsquigarrow$ section of a certain line bundle on $\mathrm{Ell}_A(pt) = \mathcal{E}$

X^A

$\cong E^{\text{rank } A}$

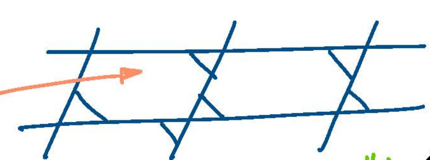
abelian variety

$\mathcal{S}$ $\rightarrow$ =  / mod $\mathbb{Z}$.

$\rightarrow$  / mod $\mathbb{Z}$

have different size, according to $\deg \mathcal{S}$

toric variety + ample line bundle $\Longleftrightarrow$ polytope  $\mathbb{P}^2$

$(\ )^2 \longrightarrow$ "periodic" toric variety  / $\mathbb{Z}^2$

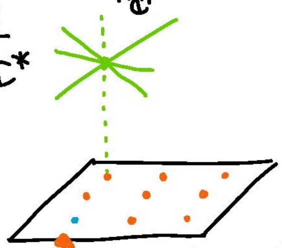
$\mathrm{rk}\, A = 1$ $\quad H^*_{\mathbb{C}^*}(X^{\mu_3})$ $\quad$ all these pieces ... $\quad K_{\mathbb{C}^*}(X)$ $\quad K_{\mathbb{C}^*}(X^{\mu_3})$

$\mathrm{rk}\, A = 1$

$A = \mathbb{C}^*$

$\cup$

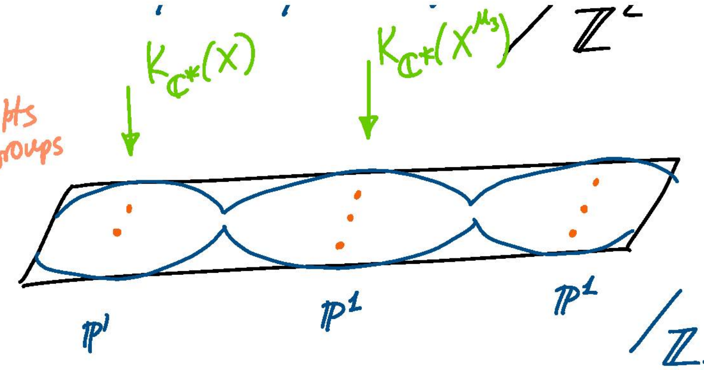
$\mu_3 = \sqrt[3]{1}$

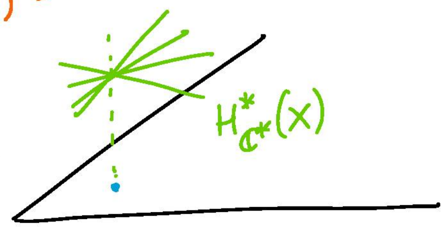


$E = \mathbb{C}^2/\Lambda = \mathrm{Ell}_A(pt)$

$E[3] = \mathrm{Ell}_{\mu_3}(pt)$

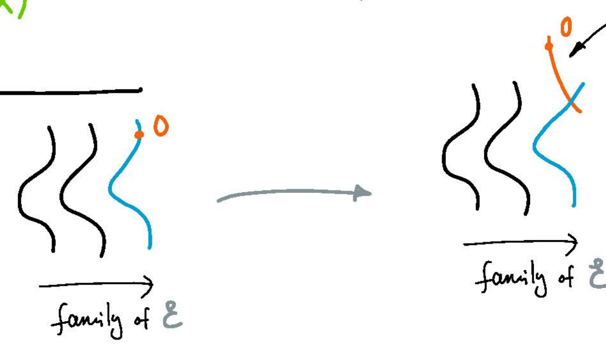
all these pieces reflect fixed pts of subgroups





everything else escaped to ∞

In other words blow up origin $\in \mathrm{Ell}_A(pt) = \mathcal{E}$ in the special fibre

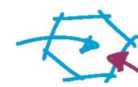


projective space

$\mathbb{P}(T_0\mathcal{E} \oplus \mathbb{C})$
$\|$
$H^\bullet_A(pt)$.

$\mathrm{Stab}(F_i)\big|_{F_j}$ for ς for α is a section of $\mathcal{S} = \Theta(T^{1/2}) \otimes$ gree zero

since sections are monomials that fit inside



Newton( ) has to fit inside

Newton$\left(\Lambda^\bullet T^{1/2}\big|_{F_j}\right)$ + shift.

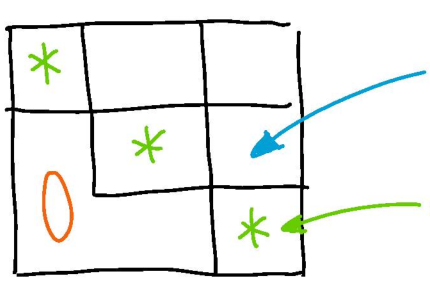
for $\angle$ degree$_\alpha$( ) < degree$_\alpha$ Euler

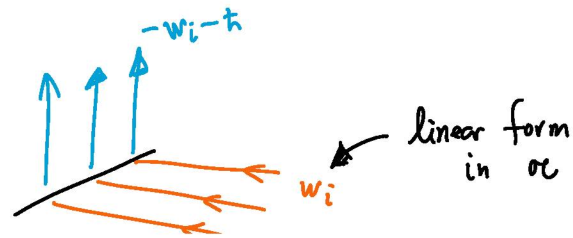
$= \frac{1}{2}$ codim F_j

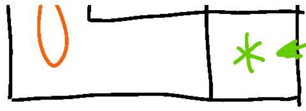
Super simple:

(existence and uniqueness for stable envelopes in $H^*_T(X)$ is rather easy. [qgqc])

Stab

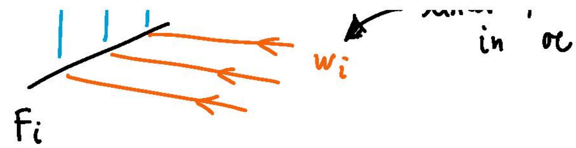




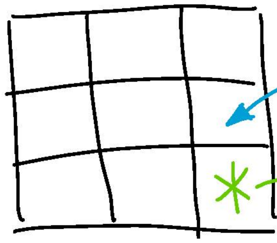


deg $= \frac{1}{2}$ codim

$= \pm \prod (w_i + \hbar)$



$\Rightarrow R(u) =$



$O(\frac{1}{u})$

$\prod \frac{w_i + \hbar}{w_i} + O(\frac{1}{u}). = 1 + O(\frac{1}{u})$

rational

because ordering of factor depends on conventions

$\boxed{R_{rat}(\infty) = 1}$ also $R\big|_{\hbar = 0} = 1$

$$\boxed{(12)\, R(-u)^{-1} (12) = R(u)}$$

$\Downarrow$

$$R(u) = 1 + \frac{\hbar}{u} r + \cdots\cdots$$

classical r-matrix

$[r, (12)] = 0.$ r is symmetric

<u>YB</u>

$$R_{12}(u_1 - u_2) R_{13}(u_1 - u_3) R_{23}(u_2 - u_3) = R_{23} R_{13} R_{12}$$

take $\frac{1}{u_1}$ coefficient as $u_1 \to \infty$: $[r_{12} + r_{23}, R_{23}(u_2 - u_3)] = 0$

in particular:

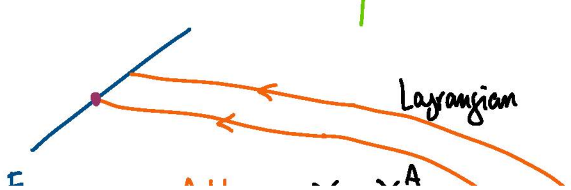
$$[r_{12} + r_{13}, r_{23}] = 0 \quad \text{and symm.}$$

$$[r_{13}, r_{23}] = -[r_{12}, r_{23}]$$

take matrix elements in 1 and 2 $\Rightarrow$ matrix elements form a Lie algebra $\mathfrak{g}$

$r \in S^2 \mathfrak{g}$ is an invariant

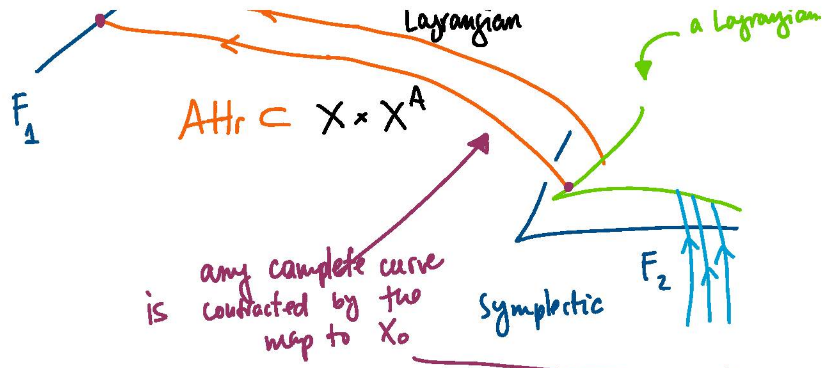
The matrix r is an example of a Lagrangian Steinberg correspondence between $X(w_1) \times X(w_2)$ and $X(w_1') \times X(w_2')$ (where $w_1 + w_2 = w_1' + w_2'$).



a Lagrangian

A correspondence

$Z \subset X \times Y$



$Z \subset X \times Y$ is Steinberg if $\exists$ affine V such that $X \to V$, $Y \to V$ proper

$$\boxed{Z \subset X \times_V Y}$$

X = Nakajima variety $= \dots /\!\!/ G$

$= \mathrm{Proj} \bigoplus_{n \geq 0} R_n \longrightarrow \mathrm{Spec}\, R_0 = X_0$ ← algebra of invariants

affine

$$X^A \xrightarrow{\text{inclusion}} X \longrightarrow \underset{\text{affine.}}{X_0}$$

In fact: Nakajima varieties are examples of symplectic resolutions, i.e. $X \longrightarrow X_0$ is birational on its image.

for such X, Steinberg variety $= X \times_{X_0} X$ is isotropic
$\Rightarrow$ its top dimensional components are Lagrangian

In particular, $\dim \text{Steinberg} \leq \dim X$

$$(1 \otimes \Delta) R = R_{12} R_{13}$$

$\Rightarrow (1 \otimes \Delta) r = r_{12} + r_{13}$, commutes not only with r_{23}, but R_{23} !

general feature of Lagrangian Steinberg correspondences, they intertwine R-matrices.

rational, that is in $H_T^{\bullet}(\dots)$