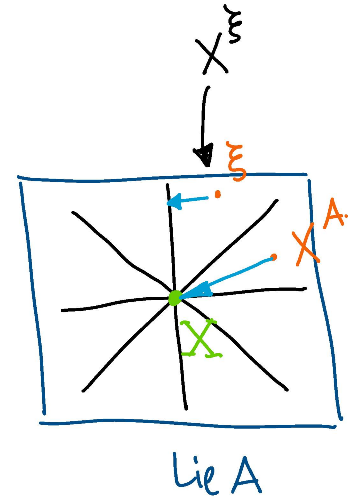
August 19, 2020

Wednesday, August 19, 2020 10:24 AM

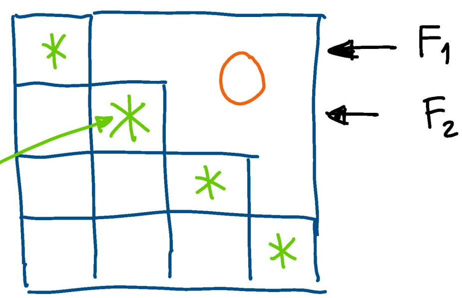
Enumerative geometry & geometric representation theory

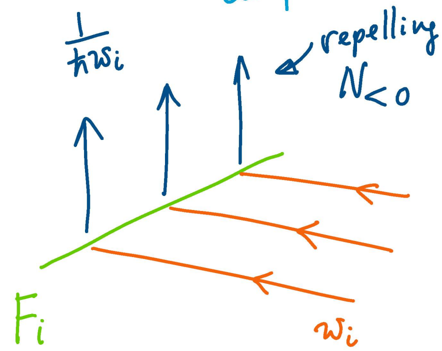
Start time Moscow 17:30
New York 10:30



$$X^A \xrightarrow{\text{Stab}} X \xrightarrow{\text{incl}^*\ \text{restriction}} X^A$$

$X^A = \bigsqcup_i F_i$ has many components





Euler class of $N_{X/X^A,<0}$

$\parallel$

$\prod \vartheta(w_i \hbar)$ nonzerodivisor on $\mathrm{Ell}_{eq}(F_i)$

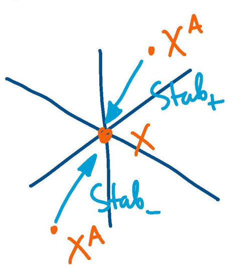
$\downarrow$

$\mathrm{Ell}_{eq}(pt)$

w_i chern roots (includes the action of A).

nontrivial

$\Rightarrow$ Stab is invertible after localization



$R = \text{Stab}_-^{-1} \circ \text{Stab}_+$ has poles precisely where $\prod \vartheta(w_i) = 0$

exists by the above

! $=$ $\text{Stab}_+^{\text{Transpose}}$, $T^{1/2} \mapsto T^{1/2}_{opp} = TX - T^{1/2}$

dynamical variable $z \mapsto z^{-1}$.

cone in Lie A, polarization, dynamical variable

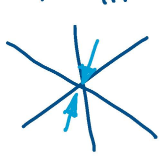
$\downarrow$

opposite

+ transpose

inverse of Stab

pictures like this



should not lead you to think in terms of reflection groups

central symmetry relates

Transpose: w.r.t. something like $(\alpha,\beta) \longrightarrow \int_X \alpha \cup \beta$

$\int$ is pushforward: $X \longrightarrow pt$ has normal bundle $-TX$

$\Theta(TX) \longrightarrow \mathcal{O}_{Ell(pt)}$.

$$\underset{\alpha^\vee}{\mathcal{S}} \otimes \underset{\beta^\vee}{\mathcal{S}^\nabla} \longrightarrow \Theta(TX)$$

$$\underbrace{\Theta(TX)\otimes \mathcal{S}^{-1}}_{\Theta(T^{1/2}_{opp})\otimes \bigotimes_i \mathcal{U}(\mathcal{L}_i, z_i^{-1})} \qquad \mathcal{S} = \Theta(T^{1/2}X)\otimes \bigotimes_i \mathcal{U}(\mathcal{L}_i, z_i).$$

Proof: $\mathrm{Stab}_-^T \circ \mathrm{Stab}_+$

with a change of parameters

here we get a section of $\Theta(TX)$

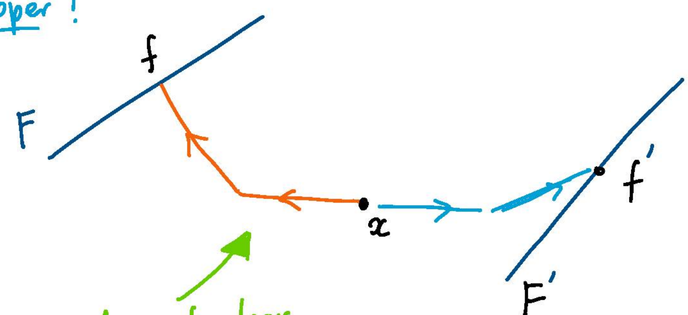
$$\underbrace{X^A \times X}_{\mathrm{Stab}_-} \times X^A, \qquad X^A \times \underbrace{X \times X^A}_{\mathrm{Stab}_+.}$$

$$\downarrow \text{push-forward}$$

$$X^A \times X^A$$

this pushforward is proper!

Suppose: $X \xrightarrow{\text{projective}} X_0$ affine



a chain of torus trajectories

a complete curve collapsed in X_0

Supp $\mathrm{Stab}^T_{-,12} \cdot \mathrm{Stab}_{+,23}$

$\subset X^A \times_{X_0} X^A$ proper. $\Longrightarrow$ $\mathrm{Stab}_-^T \circ \mathrm{Stab}$ is regular.

(no poles)

In elliptic cohomology, it is very powerful to argue by rigidity

If $\mathcal{L}$ is a degree zero line bundle on an abelian variety $\mathcal{E}$ $\quad \mathcal{L} \neq \mathcal{O} \iff H^\bullet(\mathcal{L}) = 0.$

If $\mathcal{L}$ is a degree zero line bundle on an abelian variety $\mathcal{E}$ then $\mathcal{L} \neq \mathcal{O} \iff H^\bullet(\mathcal{L}) = 0.$

if a nontrivial line bundle $\mathcal{L}$ has a section then this section $= 0$

$$\mathrm{Stab}_-^T \circ \mathrm{Stab}: \quad F_i \longrightarrow F_j$$

$\Theta(T^{1/2}_{F_i}) \otimes \mathcal{U}(\mathcal{L}_k, z_k) \longrightarrow$ similar

with a shift.

fixed by A.

$$\mathrm{Ell}_{eq}(F_i) \xrightarrow{p} \mathrm{Ell}_{eq/A}(F_i).$$

fibration in $\mathrm{Ell}_A(pt)$.

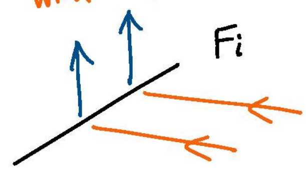
only this piece is nontrivial on the fibers of

$$\mathrm{Stab}_-^T \circ \mathrm{Stab}\Big|_{\text{fibers of } p} = \text{a section of a trivial bundle} \otimes$$

$$\otimes\, \mathcal{U}(\mathcal{L}_k\big|_{F_i}, z_k)^{-1} \otimes \mathcal{U}(\mathcal{L}_k\big|_{F_j}, z_k)$$

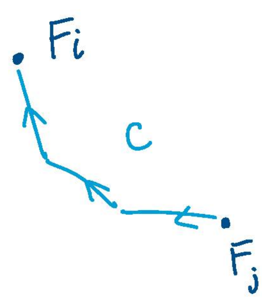
take the weight of A-action

with shifts



$F_i = F_j$ this is trivial, and by our setup this is the identity map $=$ diagonal in $F_i \times F_i$

$F_i \neq F_j$ this line bundle above is not trivial



if $\mathcal{L}$ is an ample line bundle

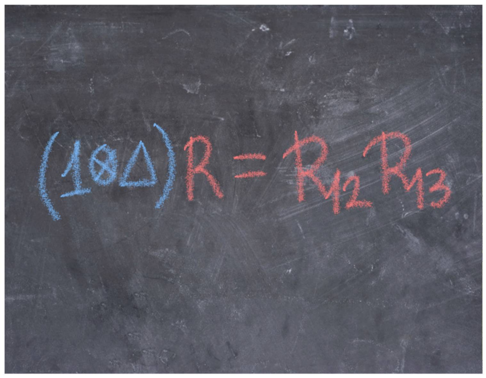
$\deg \mathcal{L}|_C > 0 \quad \xrightarrow{\text{localiz.}}$

weight $\mathcal{L}|_{F_i} >$ weight $\mathcal{L}|_{F_j}$

$\Rightarrow \exists\, \mathcal{L}$ such that this characters are not equal

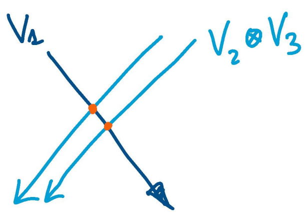
line bundle $\underset{\text{generically}}{\neq} \mathcal{O}$, section $= 0$

$$\mathrm{Stab}_-^T \circ \mathrm{Stab}_+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \square$$



Last time YB

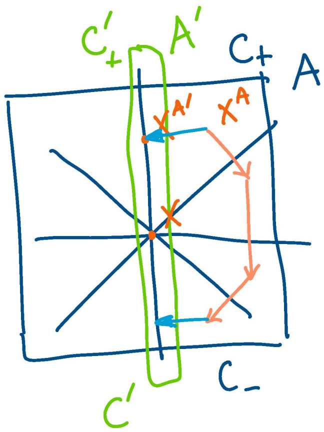
another cornerstone of quantum groups



$$R_{V_1, V_2\otimes V_3} = R_{V_1,V_2} R_{V_1,V_3}$$

$$(1\otimes\Delta)R = R_{12}R_{13}$$

is an example of the following general phenomenon: let $A' \subset A$ be a subtorus



$R_{X^{A'}}$

defined in terms of A' without A

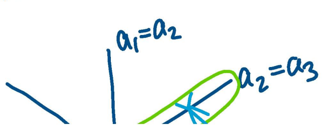
$$\begin{tikzcd} X^{A'} \arrow[d, "R_{C'_+ \to C'_-}"'] & & X^{A} \arrow[ll, "\mathrm{Stab}"'] \arrow[d, "\prod R_{\text{walls in } A}"] \\ X^{A'} & & X^{A} \arrow[ll, "\mathrm{Stab}"] \end{tikzcd}$$

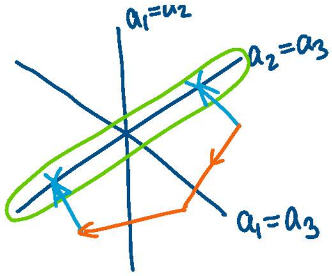
YB etc. $\quad X = X_{123}$

$X^A = X_1 \times X_2 \times X_3 \quad , \quad A_{12} \subset A \quad a_1 = a_2$

$X^{A_{12}} = X_{12} \times X_3$

$\vdots$





$$X^{A_{12}} = X_{12} \times X_3$$

$$\begin{array}{ccc} X_1 \times X_{23} = X^{A_{23}} & \longleftarrow & X_1 \times X_2 \times X_3 \\ \big\downarrow R_{1,23} & & \big\downarrow R_{12} R_{13} \\ X_1 \times X_{23} & \longleftarrow & X_1 \times X_2 \times X_3 \end{array}$$

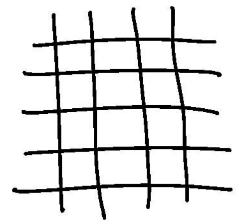
Another interesting appl:

quiver o—o—o $\rightsquigarrow$ geometric construction of $\widehat{gl}(4)$

g loops. $\longrightarrow$ quiver $\mathbb{Z}^g$

torus of rank g acts

fixed locus



$\mathbb{Z}^2$

R-matrices for this $=$ $\prod$ of R-matrices for $\mathbb{Z}^1$.

$g = 1$.

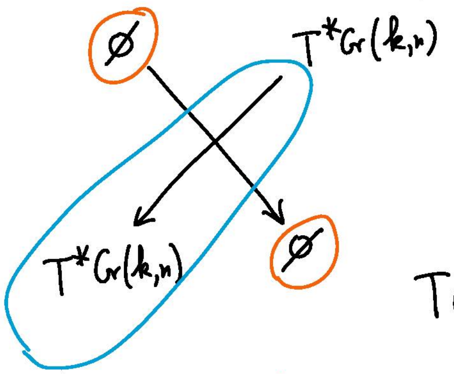
R-matrices for this $= \prod R$ for $\longrightarrow \widehat{gl}(\infty)$

$$R = \mathrm{Stab}_-^{-1} \circ \mathrm{Stab}_+ = \begin{pmatrix} * & /\!/ & /\!/ & /\!/ \\ 0 & * & /\!/ & /\!/ \\ 0 & 0 & * & /\!/ \\ 0 & 0 & 0 & * \end{pmatrix}^{-1} \begin{pmatrix} * & 0 & 0 & 0 \\ /\!/ & * & 0 & 0 \\ /\!/ & /\!/ & * & 0 \\ /\!/ & /\!/ & /\!/ & * \end{pmatrix}$$

Euler($N_{<0}$).

$$= \left(\begin{array}{c|c} & \\ \hline & \boxed{/\!/} \end{array}\right) \qquad \boxed{/\!/} = \frac{\mathrm{Euler}(N_{<0})}{\mathrm{Euler}(N_{>0})} = \prod \frac{\vartheta(\hbar w_i)}{\vartheta(w_i)}.$$

$$\Big|\ \text{[hatched box]} = \frac{\quad}{\mathrm{Euler}(N_{>0})} = \prod \frac{\quad}{\vartheta(w_i)}.$$



$$TG(n) = \bigsqcup_k T^*\mathrm{Grass}(k,n)$$

$$A = \begin{pmatrix} a & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \quad \underbrace{1 \ \cdots \ 1}_{n+1}$$

$\wr$

$TG(n+1)$

$$TG(n+1)^A = TG(1) \times TG(n)$$

$$/\!/$$

$$\mathrm{pt} \sqcup \mathrm{pt}$$

$0 \subset \mathbb{C}^1$ $\quad \mathbb{C}^1 \subset \mathbb{C}^1$

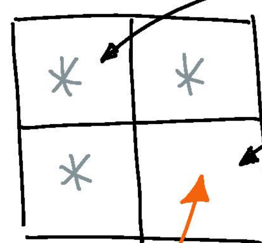
$\varnothing$

for given k

is an operator of multiplication by characteristic classes of Taut bundles

contains them all (as a function of the spectral variable a)

$T^*Gr(k-1,n)$.



$T^*Gr(k,n)$.

Euler classes of N to inside $T^*Gr(k,n+1)$

$$T(\mathrm{Grass}(k,n)) = \mathrm{Hom}(\mathcal{V}, \mathbb{C}^n/\mathcal{V})$$

$\mathcal{V} \subset \mathbb{C}^n$

$$= \mathrm{Hom}(\mathcal{V}, \mathbb{C}^n) - \mathrm{Hom}(\mathcal{V}, \mathcal{V})$$

$$T(T^*\mathrm{Grass}(k,n)) = \mathrm{Hom}(\mathcal{V}, \mathbb{C}^n) + \hbar^{-1}\, \mathrm{Hom}(\mathbb{C}^n, \mathcal{V}) - (1+\hbar^{-1})\, \mathrm{Hom}(\mathcal{V}, \mathcal{V})$$

$$T(T^*\mathrm{Grass}(k,n+1)) = \underset{N_{>0}}{a\, \mathcal{V}^*} + \underset{N_{<0}}{\frac{1}{\hbar^{-1} a^{-1}}\, \mathcal{V}} + \dots$$

$/\!/$

$\mathbb{C}^n + a\,\mathbb{C}$

if v_i are the Chern roots of $\mathcal{V}$

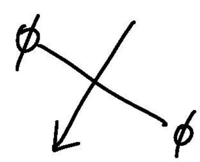
$$\frac{\vartheta(v_i/a/\hbar)}{\vartheta(v_i/a)}$$

$$YB \overset{\text{Baxter}}{\Longrightarrow} \overset{\text{commute for all } a\,!}{\mathrm{tr}_1 (z \otimes 1)\, R_{12}(a)} \qquad \text{in ordinary cohomology} \qquad \frac{v_i - a - \hbar}{v_i - a}.$$

$[z \otimes z, R_{12}] = 0.$

acts in the second factor

$z \to 0$ project onto <u>vacuum</u>, get



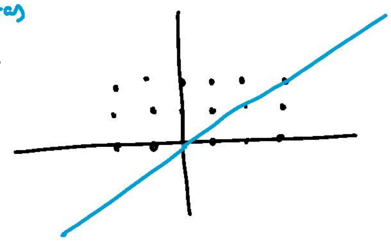
a limit of Baxter subalgebras are the algebras of <u>multiplication</u> in cohomology.

loops into Cartan $\subset E(\hat{\mathfrak{g}})$.
$\mathfrak{h} \subset \mathfrak{g}$

Many kinds of product for R-matrices:

① $$\mathcal{R}_{\mathcal{U}(\mathfrak{g})} = \overleftarrow{\prod_{\substack{\text{root subalgebras} \\ \mathfrak{g}_\alpha}}} \mathcal{R}_{\mathcal{U}(\mathfrak{g}_\alpha)}$$

$\mathfrak{g} = \widehat{\widehat{\mathfrak{gl}}}(1)$



similar for Yangians

② $$\vartheta(x) \approx \prod_n (1-q^n x)$$ regularized, really infinite

monodromy of q-difference eq. $x \mapsto qx$

also for R-matrices

(generic) flatness in elliptic curve over $\mathbb{C}$. $\Longleftarrow$

$$\left\{ \begin{array}{l} \mathcal{R}_{elliptic} = \overrightarrow{\prod} \mathcal{R}_{\mathcal{U}_q} \\ \mathcal{R}_{\mathcal{U}_q} = \overrightarrow{\prod} \mathcal{R}_{Yangian} \end{array} \right.$$

$E = \mathbb{C}^*/q^{\mathbb{Z}}$.

$\mathbb{C}^* = \mathbb{C}/2\pi i\mathbb{Z}$.

③ $$\mathcal{R}_{Quiver/\Gamma} = \overrightarrow{\prod_{\Gamma}} \mathcal{R}_{Quiver}$$

here

locally finite, of the same type

ell $\longleftrightarrow$ ell

$\mathcal{U}_q \longleftrightarrow \mathcal{U}_q$

Yang $\longleftrightarrow$ Yang.