August 12, 2020
Wednesday, August 12, 2020 9:57 AM

Enumerative geometry & geometric representation theory

Start time Moscow 17:30
New York 10:30

R- matrices, YB equations, etc.

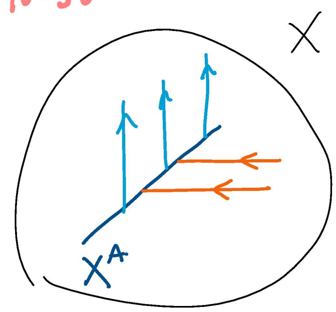
$X^A \rightarrow X \leftarrow X^A$

$a_1 = a_2$

Lagrangian correspondence

improved version of attracting manifolds

could be $X_1 \times X_2$



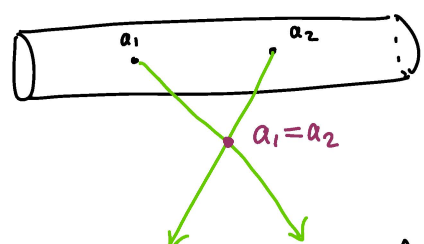
$\mathrm{Lie}_{\mathbb{R}} A$
$\cong$
$a \in \mathbb{C}^*$

acting on cohomology give

$(a_1, a_2) \in (\mathbb{C}^*)^2 \supset$ diag $\mathbb{C}^*$ acts trivially

$(\mathbb{C}^*)^2$ 2 points in $\mathbb{C}^*$

$V_1(a_1) \otimes V_2(a_2) \xrightarrow{\mathrm{Stab}_-} V \xleftarrow{\mathrm{Stab}_+} V_1(a_1) \otimes V_2(a_2)$
opp

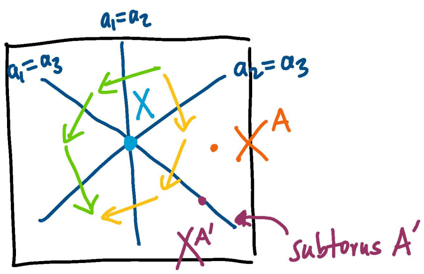


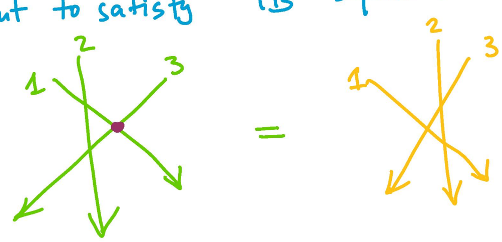
$$R(a_1/a_2) = \mathrm{Stab}_-^{-1} \circ \mathrm{Stab}_+$$

R- matrix

$A = (\mathbb{C}^*)^3 / \mathrm{diag}\, \mathbb{C}^*$

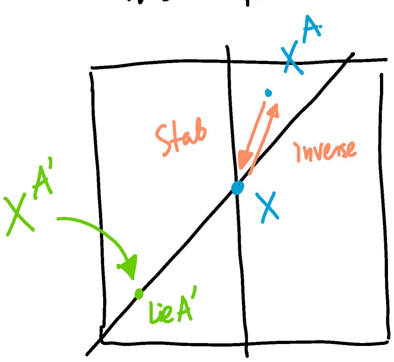
want to satisfy YB equation

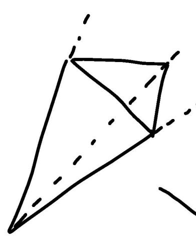


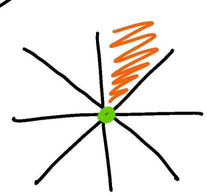


This setup, with chambers in Lie A is more general than

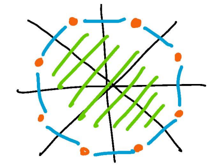
specific to root systems of type $A_1, A_2, \ldots$



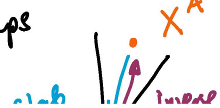




dual polytope



for every pair of adjacent cones or faces we have maps



$A' \subset A''$

Thm "The triangle lemma"

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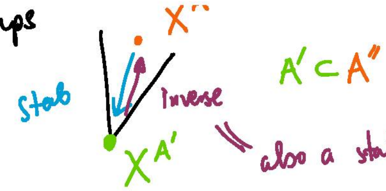
these maps compose

$\uparrow$ i.e. form a representation of the corresponding groupoid

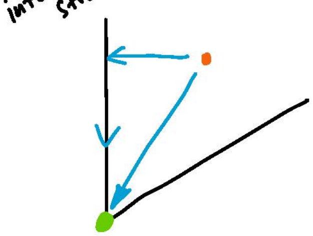
maps



$A' \subset A''$

$\Vert$ also a stable envelope

intermediate stratum

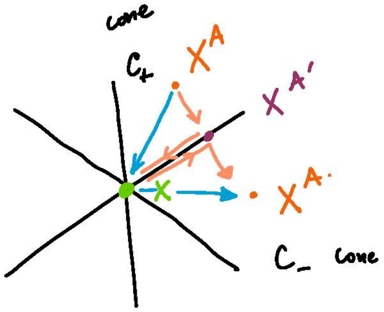


$A' \subset A''' \subset A''$

$$\begin{tikzcd} X^{A''} \arrow[rr, "\text{Stab for } A'/A''"] \arrow[dr, "A'''/A''"'] & & X^{A'} \\ & X^{A'''} \arrow[ur, "A'/A'''"'] & \end{tikzcd}$$

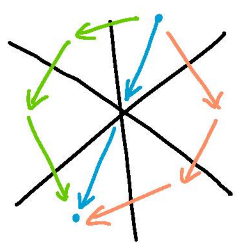
needs a precise statement

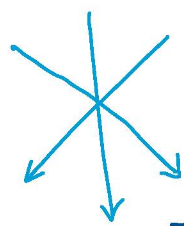
YB



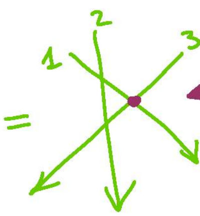
$$R_{C_+ \to C_-} = \mathrm{Stab}_{C_-}^{-1} \circ \mathrm{Stab}_{C_+}$$

$=$ can be computed inside X^A





$=$



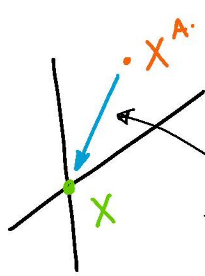
is computed inside $V_1 \otimes V_3$

fixed locus for the corresponding subtorus.

YB is proven by defining and showing $* = *$

In fact, we should be proving the <u>dynamical</u> YB equation

Remind the setup



want a "Lagrangian" elliptic correspondence inside

$X \times X^A$

$\uparrow$ section of a certain line bundle on $\mathrm{Ell}_{eq}(X \times X^A)$

zero bundle

$1 = \text{identity} = [X] = \ldots\ldots$ a section $\mathcal{O}_{\mathrm{Ell}_{eq}(X)} = \Theta(0)$

$[pt] =$ pushforward 1 under $pt \to X$

is a section $\Theta(TX)$

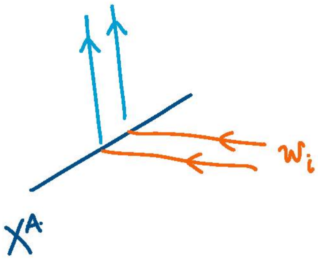
would be like the canonical ...

is a section $\Theta(TX)$ ← would be like the canonical of $\mathrm{Ell}_{eq}(X)$ if X were compact

Lagrangians = sections of $\mathcal{S}$ $\approx \Theta(TX)^{1/2} \otimes$ degree zero

unavoidable

$\frac{1}{w_i \hbar}$ chern roots of $N_{X/X^A,<0}$



chern roots of $N_{X/X^A,>0}$.

$\hbar$ = weight of the symplectic form of X.

$$[\mathrm{Attr}] = \prod \vartheta(w_i \hbar) \qquad [\mathrm{Repell}] = \prod \vartheta(w_i)$$

not sections of the same bundle, $\prod \frac{\vartheta(w_i\hbar)}{\vartheta(w_i)}$ ← has degree 0 in $a \in \mathrm{Ker}\,\hbar$

Assume X has a polarization $T^{1/2}X$, i.e. a solution to

$$[T^{1/2}X] + [(T^{1/2}X)^\vee] \equiv [TX]$$

in A-equivariant K-theory

$$\mathcal{S} = \Theta(T^{1/2}X) \otimes \mathcal{U}$$

$\mathcal{U} = \bigotimes \mathcal{U}(\mathcal{L}_i, z_i)$ — $\mathcal{L}_i$: basis in $\mathrm{Pic}(X)$; z_i: dual coordinates

has a section $\frac{\vartheta(s_i z_i)}{\vartheta(s_i)\vartheta(z_i)}$, $s_i = c_1(\mathcal{L}_i)$

X^A has its own polarization $T^{1/2}_{X^A} = \left(T^{1/2}_X\Big|_{X^A}\right)^A$

$T^{1/2}_X\Big|_{X^A}$ = fixed + attracting + repelling

in addition has index $= T^{1/2}\Big|_{X^A,>0}$

if $X = T^*M$, $T^{1/2}_X = TX$

Stable envelope:

$$J_A \longrightarrow J$$

target

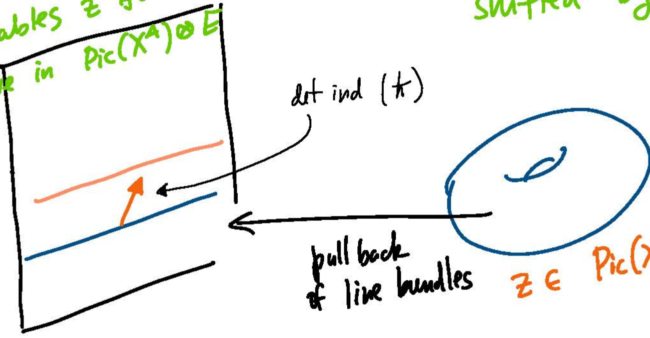
$\parallel$

$$\underbrace{\ominus(T^{1/2}_{X^A}) \otimes \mathcal{U}\big|_{X^A}}$$

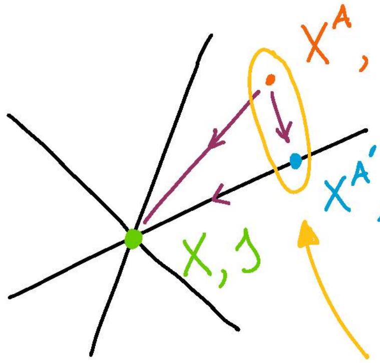
shifted by $z \mapsto z - \hbar \underbrace{\det \mathrm{ind}}$

a line bundle on X^A.
i.e. $\mathrm{Pic}(X^A)$
so, a cocharacter of
$\mathrm{Pic}(X^A) \otimes \mathbb{C}^*$
where the Kähler variables z live for X^A

variables z for X^A live in $\mathrm{Pic}(X^A)\otimes E$



one can plug $\hbar$ into any cocharacter, in
in particucal det ind



X^A, $J_A(-\hbar \det \mathrm{ind}_{X/X^A})$

$X^{A'}$, $J_{A'}(-\hbar \det \mathrm{ind}_{X/X^{A'}})$.

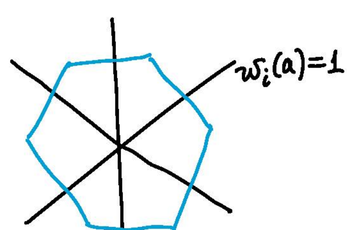
commutes by uniquness of stable envelopes

is the stable envelope for X^A inside $X^{A'}$ shifted by

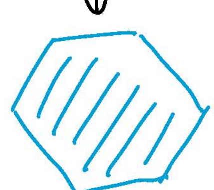
YB equation will have explicit shifts in Kähler (a.k.a. dynamical) variables

Exercise: write this out for T^*Gr or general Nakajima variety

Lie A



$(\mathrm{Lie}\ A)^*$
$\cup$

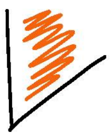


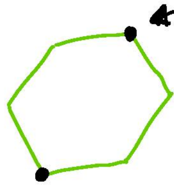
Newton polytope for $\Lambda^\bullet N_{X/X^A}$ — $T^{1/2}$
projection of a cube of dimension = rank.

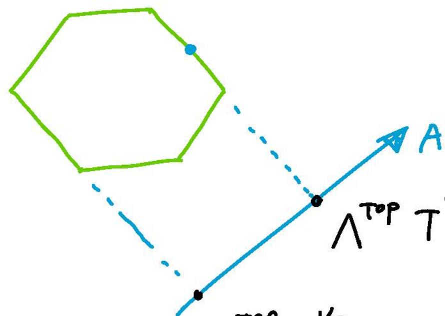
where w_i are the

where w_i are the weights in N_{X/X^A}.

but really they are Chern roots of a bundle, i.e. this zonotope lives not just in $(\mathrm{Lie}\,A)^*$. but in fact in $\mathrm{Pic}_A(X^A)$.

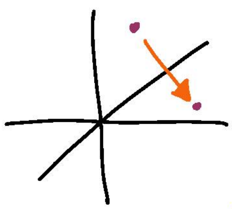


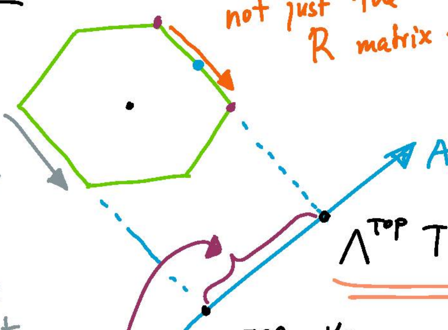




not just weights also elements of $\mathrm{Pic}_A(X^A)$

line bundles





not just the R matrix for X^A inside $X^{A'}$, it remembers the ambient geometry throug. this shift.

almost the same, but not quite.

is the dynamical shift.

fixed by topology

depends on concrete situation.

polytope is fixed by topology up to translations

$=$.

$$Y \xrightarrow{f} X \leftarrow \mathcal{V}$$

$$f_{\circledast} : \Theta(-N_f) \longrightarrow \mathcal{O}$$

$$\Theta(\underbrace{-N_f + f^*\mathcal{V}}_{0}) \longrightarrow \Theta(\mathcal{V})$$

$Y = pt.$

$\mathcal{V} = TX$

$$\vartheta \longrightarrow \Theta(TX)$$