August 5, 2020

Wednesday, August 5, 2020 9:38 AM

Enumerative geometry & geometric representation theory

Start time Moscow 17:30

New York 10:30

for $\widehat{\mathfrak{gl}}_n$ $\bigotimes$ evaluation fund $= \Lambda^{k_i}\mathbb{C}^n(a_i)$ realized geometrically as cohomology of Nakajima quiver varieties

q-quantum elliptic ...

with quiver $\underbrace{\bullet\!-\!\circ\!-\!\circ\!-\!\circ}_{n-1 \text{ nodes}}$ A_{n-1}

stable envelopes $\Longrightarrow$ explicit formulas for offshell Bethe eigenfunctions

$\widehat{\mathfrak{gl}}_2$, quiver $\bullet$ (no arrows)

$$\bigotimes_{i=1}^{n} \mathbb{C}^2(a_i) \rightsquigarrow TG(n) = \bigsqcup_{k=0}^{n} T^*\mathrm{Grass}(k,n) \circlearrowleft$$

$$A = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \subset GL(n)$$

$TG(n)^A$ = coordinate subspaces $\subset$ Grass $\subset T^*$Grass.

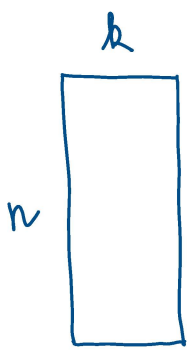
2^n

$$\# \; T^*\mathrm{Grass}(k,n)^A = \binom{n}{k}$$

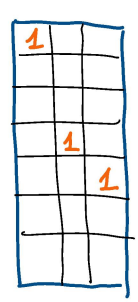
$$\mathrm{Gr}(k,n) = \{ V \subset \mathbb{C}^n \}, \quad \dim V = k$$

V = image of $\mathbb{C}^k \longrightarrow \mathbb{C}^n$

$$\mathrm{Gr}(k,n) = \mathrm{Hom}(\mathbb{C}^k \to \mathbb{C}^n)_{\text{rank } k} / GL(k)$$



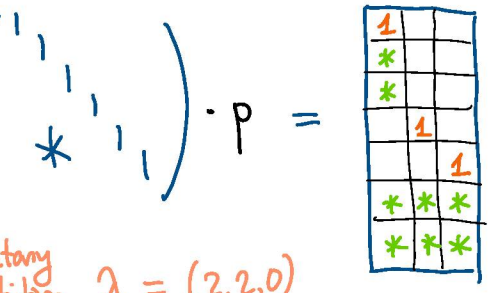
of rank k, modulo column operations $\supset$



$= \mathrm{Gr}(k,n)^A$

$\mathbb{P}^8$ Schubert

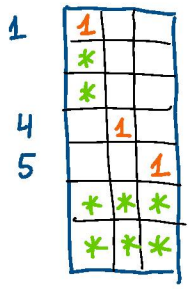
$$\mathrm{Attr}(p) = \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & * & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{pmatrix} \cdot p = \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \end{array}$$

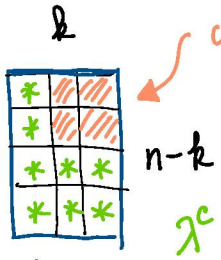


$\simeq \mathbb{C}^8$ Schubert cell

arbitrary

$\mathcal{G}_p = \overline{\mathrm{Schubert}(p)}$





complementary partition $\lambda = (2,2,0)$

$\lambda + \rho = (4,3,0)$.

$|\lambda|$ = codimension of $\mathcal{G}_p$

degree in cohomology

$|\lambda^c|$ = dimension of $\mathcal{G}_p$

λ^c

partition $(4,2,2) \subset$  k.

$$H^*(Gr) = \langle \text{Chern classes of the tautological bundle } V \rangle = \text{Symm functions in } \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_k \end{pmatrix} \Big/ \text{ideal}$$

$\mathfrak{gl}(k)$

¿ what is the class of $\mathcal{G}_\lambda$?

<u>Answer:</u> Schur function $s_\lambda(x_1,\dots,x_k) = \mathrm{Symm} \dfrac{\prod x_i^{\lambda_i+\rho_i}}{\prod_{i<j}(x_i - x_j)}$ $\qquad \rho = (k-1, k-2, \dots, 0)$.

positions of 1

a polynomial in x !

$m_\ell(x) = x^\ell$ $\qquad a_i = 0$

or

$m_\ell(x) = (x-a_1)(x-a_2)\dots(x-a_\ell)$ $\qquad m_\ell(a_m) = 0$ unless $m > \ell$

Newton interpolation

replace by any generalization of a monomial

e.g. $\prod_i m_{\lambda_i+\rho_i}(x_i)$

arbitrary collection of functions of 1 variable

$$s_\lambda^{\text{interp.}} = [\mathcal{G}_\lambda] \text{ in } H_A^*(Gr)$$

Lascoux – Schützenberger

<u>Proof:</u> 1) has degree $|\lambda|$

2) vanishes at all fixed pts that are not in $\mathcal{G}_\lambda$

$$s_\lambda^{\text{interp}}(a_\mu) = 0 \quad \text{unless} \quad \lambda \subset \mu$$

$(a_{\mu_1+\rho_1}, a_{\mu_2+\rho_2}, \dots)$

characterization (up to multiple).

$=$

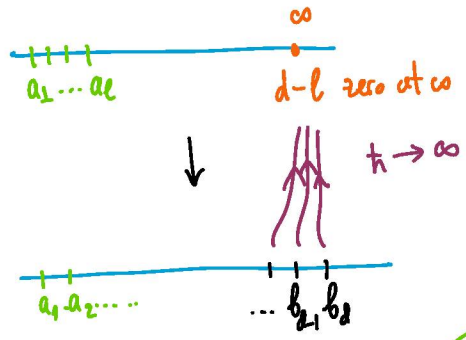
the unique (up to mult.) poly of degree ℓ

$$m_\ell(x) = (x-a_1)(x-a_2)\dots(x-a_\ell)$$

the unique poly of degree ℓ that vanishes at $a_1, \dots, a_\ell$

has a pole of a certain order at $x=\infty$

$\mathcal{O}(d)$ on $\mathbb{P}^1$



$a_d + \hbar$

$$m_\ell(x) = \underbrace{(x-a_1)(x-a_2)\dots}_{\ell} \underbrace{\dots (x-b_{\ell-1})(x-b_\ell)}_{d-\ell}$$

with this specialization one can do more

$$\frac{1}{\hbar^k} \mathrm{Symm} \frac{\prod_{i=1}^n m_{\ell_i}(x_i)}{\prod_{i<j}(x_i-x_j)(x_j-x_i+\hbar)} = \frac{1}{\prod_{i,j} x_i - x_j + \hbar} \mathrm{Symm} \prod_{i<j} \frac{x_i - x_j + \hbar}{x_i - x_j} \prod_{i=1}^n m_{\ell_i}(x_i).$$

rational function

$\hbar$ scales

Hall-Littlewood-type function

stable envelope for $T^*Gr(k,n)$ has degree $k(n-k) = \frac{1}{2}\dim$

become Schubert classes $\mathfrak{S}_\lambda$ when $\hbar \to \infty$

polynomial push-forward from $\mu^{-1}(0)$ to the whole stack

$$T^*Gr \underset{\text{open}}{\subset} \mu^{-1}(0) \underset{\text{closed}}{\subset} \left[\mathrm{Hom} \oplus \mathrm{Hom} / GL(k) \right]$$

a regular function

equivariant weight of the moment map

$a_1, \dots, a_n$ $\quad x_1, \dots, x_k$ $\quad H \subset GL(k)$

$$\mathrm{Spec}\, H^*(T^*Gr) \underset{\text{closed}}{\subset} \mathrm{Lie}\, A \times \mathrm{Lie}\, H / W$$

$$\| $$

$$\bigcup \mathrm{Lie}\, A$$

fixed pts

by chern roots of the tautological bundle

$$T^*Gr = \left\{ \begin{matrix} \mathrm{Hom}(\mathbb{C}^k \xrightarrow{I} \mathbb{C}^n) \oplus \\ \mathrm{Hom}(\mathbb{C}^n \xrightarrow{J} \mathbb{C}^k) \end{matrix} \right\} / GL(k).$$

$J\hbar^{-1}$

$\mathfrak{gl}(k) \ni JI = 0$

moment map

weight $\hbar^{-1}$

I injective

or J surjective

weight $\hbar^{-1}$

I injective or J surjective

Elliptic version:

① take elliptic envelopes for $T^*\mathbb{P}^{n-1}$ as our functions m_ℓ

(if $k=1$ $T^* Gr(k,n)$).

$$\underset{\substack{\| \\ x^{-1}}}{m_\ell(s)} = \underset{a_1 - x}{\vartheta(a_1 s)} \cdots \underbrace{\frac{\vartheta(a_\ell s z \hbar^{n-\ell})}{\vartheta(z\hbar^{n-\ell})}}_{\text{new piece with dynamical variable.}} \cdots \vartheta(a_n \hbar s).$$

② $$\mathrm{Symm} \frac{\prod m_\ell(x_i)}{\prod_{i<j} \vartheta(x_i/x_j)\vartheta(x_j/x_i\hbar)}$$ actual elliptic cohomology class

push forward to $[\mathrm{Hom} \oplus \mathrm{Hom} / GL(k)]$

$$\mathrm{Symm} \prod_{i<j} \frac{\vartheta(x_i/x_j\hbar)}{\vartheta(x_i/x_j)} \prod m_\ell(x_i)$$

Proof $\Longleftarrow$ D. Shenfeld (Phd thesis).

$$\begin{tikzcd} \text{"} \quad \mu_G^{-1}(0)/H \arrow[r] \arrow[d] & \mu_H^{-1}(0)/H \quad \text{"} \\ \mu_G^{-1}(0)/G & \end{tikzcd}$$

hypertoric variety if $G \circlearrowright$ vector space such as $\mathrm{Hom} \oplus \mathrm{Hom}$

push-forward involves $\frac{1}{\theta(x_i/x_j)}$

"smooth Lagrangian correspondence"

elliptic envelopes explicit.

$G = GL(k) \supset H =$ maximal torus

inverting this involves $\frac{1}{\theta(x_i/x_j\hbar)}$.

off-diagonal weights of the moment map.

Why only $\prod_{i<j}$? Have to look into actual geometry

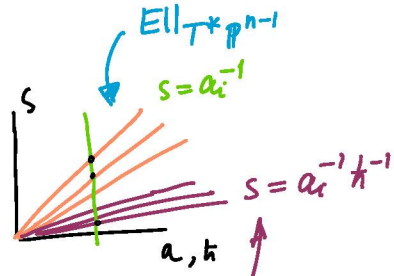
e.g. $GL(k)/H$ = general conjugacy class $\simeq T^*$ Flag

Very general

e.g. for $\mathrm{Hilb}(\mathbb{C}^2, k)$ formulas in [Smirnov]

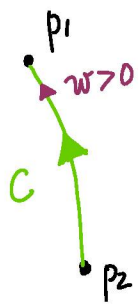
For
$$\begin{tikzcd} \mu_H^{-1}(0)/G \arrow[r] \arrow[d] & (T^*\mathbb{P}^{n-1})^k. \\ T^*Gr(k,n) & \end{tikzcd}$$
me

$\mathrm{Ell}_{T^*\mathbb{P}^{n-1}} \subset$



for the flop.

elliptic stable envelopes are flop-invariant and vanish outside of the full attracting set



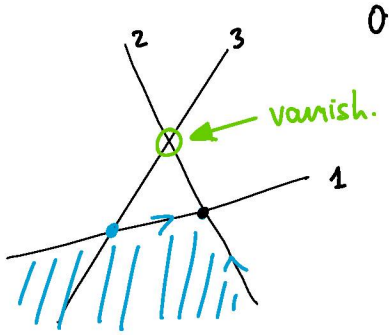
attracting order of fixed pts is reversed under flop!

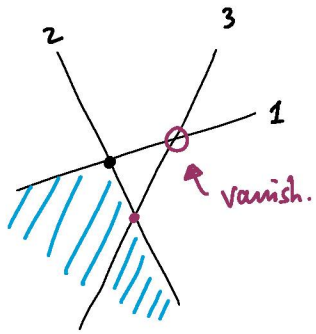
$\Longrightarrow$ section of line bundle of degree n in S vanish

$(\deg_S = n s^2)$

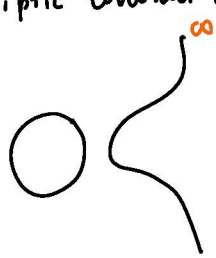
$\mathcal{O}(1)$ is an ample bundle $\langle C, \mathcal{O}(1)\rangle \overset{\text{loc}}{=} \dfrac{\text{weight } \mathcal{O}(1)_{p_1} - \text{weight } \mathcal{O}(1)_{p_2}}{w} > 0$

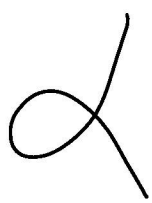
0 $\quad$ positive



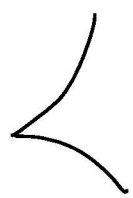


Elliptic cohomology is defined





equivariant K-theory



equivariant cohomology.

irreducible, genus 1, + smooth point

nodal

$\simeq \mathbb{P}^1/_{2\,pts} \supset \mathbb{C}^*$

cuspidal

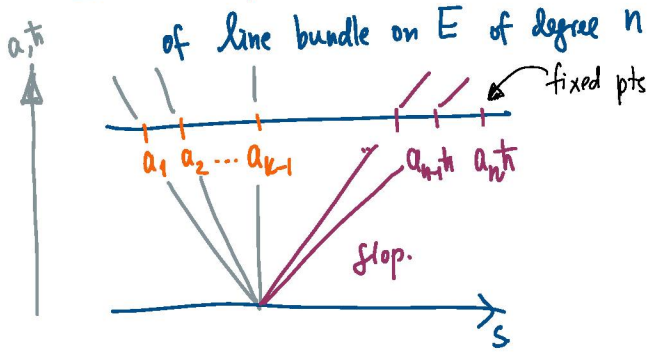
$s \in E$ $\quad$ chern root of the $\mathcal{O}(1)$ ...

E becomes $\mathbb{C}^*$

$\mathbb{C} = \mathrm{Pic}_0(\text{cuspidal})$.

$s \in E$ ← chern root of the $O(1)$

Stable envelope for $T^*\mathbb{P}^{n-1}$ section of line bundle on E of degree n



E becomes $\mathbb{C}^*$

chern root for $O(1)$ in $K_{eq}(T^*\mathbb{P}^{n-1})$.

$\mathbb{C} = \mathrm{Pic}_0(\text{cuspidal})$.

s = chern root of $O(1)$

<u>Fundamental</u> Everything we do is <u>flat</u> in the elliptic curve.