July 29, 2020

Wednesday, July 29, 2020 9:31 AM

Enumerative geometry & geometric representation theory

Start time Moscow 17:30

New York 10:30

$$X = T^* \mathbb{P}^{n-1}$$

$$\mathbb{P}^{n-1} = (\mathbb{C}^n \setminus 0) / \underset{\substack{\cap \\ s}}{\mathbb{C}^*}$$

stability condition

$$T^*\mathbb{P}^{n-1} = \{ \underset{\substack{x_1,\dots,x_n}}{x} \in \underset{\substack{\wr\wr \\ \mathbb{C}^n}}{V},\ y \in V^*,\ x \cdot y = 0,\ x \neq 0 \} / \mathbb{C}^*$$

moment map for

$$\underset{\substack{\| \\ X}}{T^*\mathbb{P}^{n-1}} \underset{\text{open}}{\subset} \{ x \in \underset{\substack{\wr\wr \\ \mathbb{C}^n}}{V},\ y \in V^*,\ x \cdot y = 0 \} / \mathbb{C}^* = T^*[V/\mathbb{C}^*]$$

moment map

quotient stack

$$\mathrm{Ell}_{eq}(X) \subset \mathrm{Ell}_{eq}(\text{stack}) = \mathrm{Ell}_{eq \times \underset{s}{\mathbb{C}^*}}(pt) = \underset{a_1,\dots,a_n}{E^n} \times \underset{\hbar}{E} \times \underset{s}{E}$$

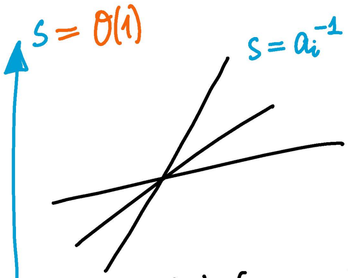
$$\underset{\substack{\| \\ GL(V) \times \mathbb{C}^*}}{eq}$$

acts on V an V^*

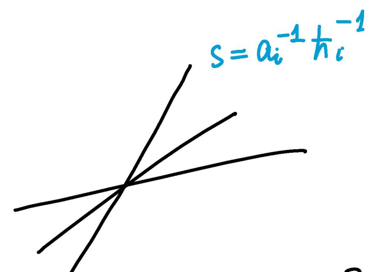
$$\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$$

scales y by $\hbar^{-1}$

$\hbar$



$$\bigcup_i \{ s a_i = 1 \}$$



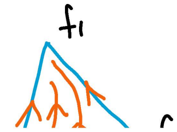
$$\bigcup_i \{ s a_i \hbar = 1 \}$$

same, but $y \neq 0$

also in $\mathrm{Ell}_{eq}(\text{stack})$ sits $\mathrm{Ell}_{eq}(X_{flop})$

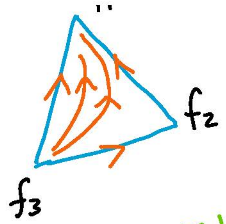
A- fixed pts in X, $\left\{ \begin{array}{l} y = 0 \\ x = (0, \dots, \underset{\substack{\uparrow \\ k}}{1}, \dots, 0) \end{array} \right\} = f_k$

$$f_1 \leftarrow f_2 \leftarrow f_3 \leftarrow \dots\dots$$



$$f_1 \leftarrow f_2 \leftarrow f_3 \leftarrow \cdots\cdots$$

k



$\mathfrak{g}$

$$\mathrm{Stab}(f_k) = \text{section of some line bundle on } \mathrm{Ell}_{eq}(X) \xleftarrow{\text{restricted}} \mathrm{Ell}_{eq}(\text{stack})$$

s on the stack

$$\Theta(T^{1/2}X) \otimes \mathcal{U}(\mathcal{O}_X(1), z)$$

Universal, or Poincaré bundle, with sections like

$$\frac{\vartheta(zs)}{\vartheta(z)\,\vartheta(s)}$$

$$TX = T^*V - \underbrace{\mathbb{C}}_{\text{moment map}}^{=\mathfrak{g}^*} - \mathbb{C}^{=\mathfrak{g}}$$

for the quotient

$$T^{1/2}X = V - \mathbb{C}$$

does not affect Θ

$\mathfrak{g} = \mathrm{Lie}(\mathbb{C}^*)$ $\ni s$

$$\mathrm{Stab} \approx \underbrace{\prod \vartheta(a_i s)}_{\Theta(V)} \times \frac{\vartheta(zs)}{\vartheta(z)\,\cancel{\vartheta(s)}}$$

supported on the full attracting set of f_k.

degree n in s
degree 1 in a_i

can't have denominators in s

can have denominators in z and $\hbar$ ← resonant loci

<u>Answer:</u>

$$\mathrm{Stab}(f_k) = \frac{\vartheta(a_k s z \hbar^{n-k})}{\vartheta(z\hbar^{n-k})} \prod_{i<k} \vartheta(a_i s) \prod_{i>k} \vartheta(a_i \hbar s)$$

weight of coordinate x_i

(weight of y_i)$^{-1}$

<u>Check:</u>

$$\mathrm{Stab}(f_k)\Big|_{f_k} = 1 \prod_{i<k} \vartheta(a_i/a_k) \prod_{i>k} \vartheta(a_i/a_k \cdot \hbar)$$

$s = a_k^{-1}$.

repelling weights at f_k.

0 by the moment equation $x \cdot y = 0$

attracting

y

repelling

$$x = \left[\boxed{0, \ldots, 0}, 0, \boxed{0, \ldots, 0}\right]$$

$$\prod \vartheta(a_i s) \prod \vartheta(a_i \hbar s)$$

$$f_k = \begin{bmatrix} \boxed{0,\dots\dots,0}, & 0, & \boxed{0,\dots,0} \\ \boxed{0,\dots\ \ 0}, & 1, & \boxed{0,\dots\ 0} \end{bmatrix}$$

repelling x attracting

$$\prod_{i<k} \vartheta(a_i s) \prod_{i>k} \vartheta(a_i \hbar s)$$

$$\Downarrow$$

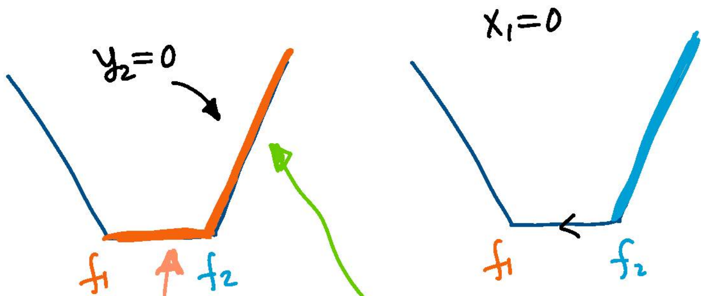
sets $x_1 = \dots = x_{k-1} = 0 \quad y_{k+1} = y_{k+2} = \dots = y_n = 0$

has the correct support

the remaining factor $\dfrac{\vartheta(a_k s z \hbar^{n-k})}{\vartheta(z \hbar^{n-k})}$ is uniquely reconstructed from being a section of the required line bundle, i.e. from the "factors of automorphy"

e.g.

$$X = T^* \mathbb{P}^1.$$



$\begin{bmatrix} y_1 & 0 \\ x_1 & x_2 \end{bmatrix}$ $\quad x_1 y_1 = 0.$

$\begin{bmatrix} 0 & 0 \\ * & * \end{bmatrix} \qquad \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$

!!! same formula works for X_{flop}

$T^* \mathbb{P}^1 \longrightarrow T^* \mathbb{P}^n \longrightarrow$ hypertoric varieties

$\downarrow$

elliptiptic $\mathfrak{sl}(2)$

$\downarrow$

elliptic $\mathfrak{gl}(n)$

Torus in $GL(V)$

$$X = T^*V /\!/\!/\!/ S$$ with some stability condition.

Assume smooth.

put $\mu_S^{-1}(0) /\!/ S$

$$1 \to S \to \begin{matrix}\text{maximal torus}\\ \text{of } GL(V)\end{matrix} \to A \to 1.$$

a fixed pt of S has the form (exchanging directions in V and V^*)

$\hbar$ acts only here

$$\begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 0\,0\,0\,0 & 0\,0\,0\,0\,0 \\ \underbrace{*\,*\,*\,*}_{V_1} & \underbrace{\boxed{0\,0\,0\,0\,0}}_{V_2} \end{bmatrix}$$

$T^{1/2}X$ polarization

free S orbit

repelling for A

$(s_1, .., s_r)$ = coordinates on S

sets $x_{r+1} = \dots = 0$.

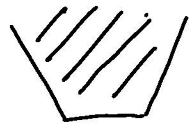
$$\text{stab}(\text{fixed pt}) = \frac{\Theta(V_1)\big|_{s=sz}}{\prod \vartheta(z_i)}\,\Theta(V_2)$$

$sz = (s_1 z_1, \dots, s_r z_r)$

no $\hbar$ here like for the last point for $T^*\mathbb{P}^{n-1}$

$\text{Pic}(X) = \text{cocharacters}(S)$

$\text{Pic}(X) \otimes \mathbb{C}^* \ni (z_1, \dots, z_r)$



toric polytope for $(a, \hbar)$.

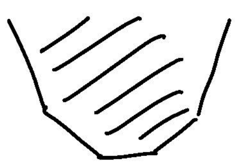
How to draw a hypertoric variety?

$\mathfrak{a}^*$

like a tesselation.

$T^*\mathbb{P}^1$

A_n- surface

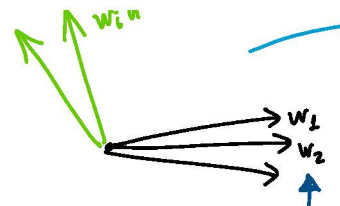


at a fixed pt.

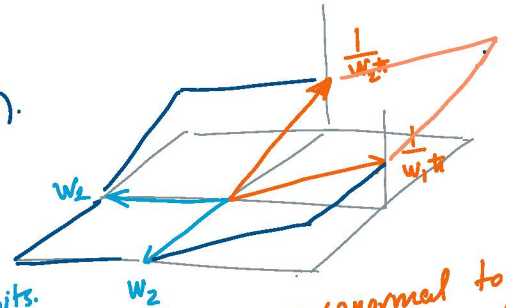
X $\frac{1}{w_i \hbar}$ dual directions

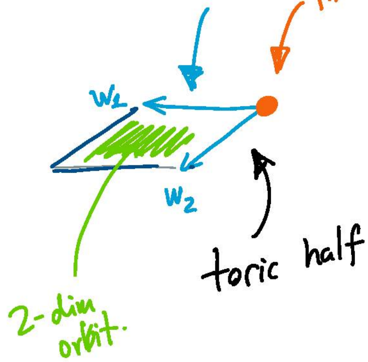
$$\xrightarrow{\text{moment map}} \text{Cone}\left(w_i, \frac{1}{w_i \hbar}\right) \in \mathfrak{a}^* \oplus (\text{Lie}\, \mathbb{C}^*_\hbar)^*.$$

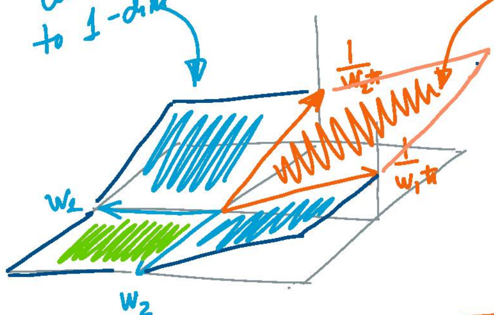
X



Cone $(w_i, \overline{w_{i\hbar}})$



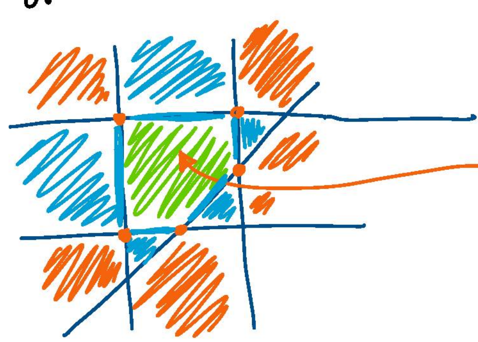




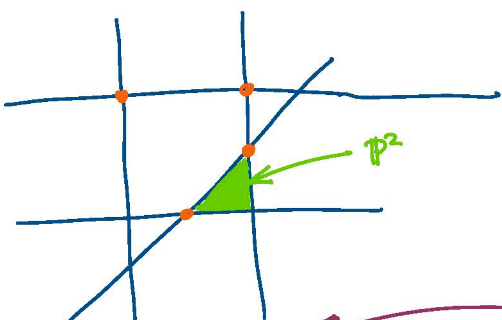
Lagrangian conormals to toric strata.

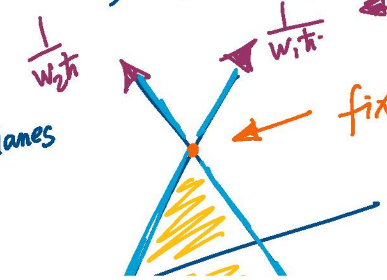
singular values of the moment map

General hypertoric variety = hyperplane arrangement in $\mathfrak{a}^*_{\mathbb{R}}$.



Blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at (∞, ∞).





these coordinates we set to 0.

5 hyperplanes
$\Downarrow$
$\dim V = 5$

hyperplanes that meet this point correspond to [illegible]/repelling [illegible] at f

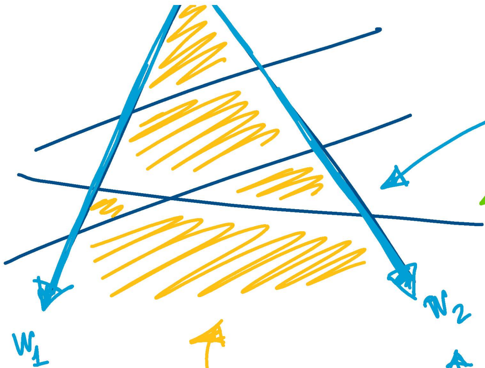
$\Downarrow$

$\dim V = 5$

$\dim A = 2$ because we are in the plane.

$\operatorname{rk} S =$

$\operatorname{rk} \operatorname{Pic}(X) =$
$= 5-2=3$



hyperplanes meet this point correspond to attracting/repelling weights at f

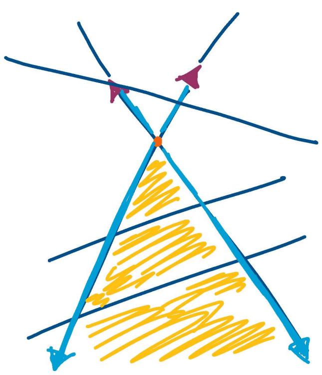
attracting

support of the stable envelope.

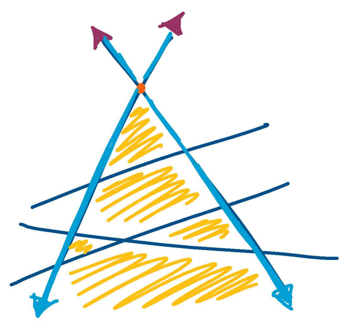
$\Theta(V_2)$ part of the formula.

hyperplanes that don't meet f $\Longrightarrow$ divisors in X span $\operatorname{Pic}(X)$

vary the variables z in $\Theta(V_1)\big|_{S=Sz}$ we get $\operatorname{Pic}(X)\otimes_{\mathbb{Z}} E$



X_{flop}



X

$TG(n) = \bigsqcup T^*Gr(k,n)$ $\quad = \quad \widehat{\mathfrak{gl}(2)} \quad =$

$$TG(n) = \bigsqcup_k T^* Gr(k,n)$$

in the sense of geom. rep theory $\qquad \widehat{\mathfrak{sl}}(2)$

$$TG(n) = \text{"} TG(n_1) \otimes TG(n_2) \text{"} \qquad n_1 + n_2 = n.$$

$TG(1) = 2\,\text{pts} \rightsquigarrow \mathbb{C}^2$ module over $\widehat{\mathfrak{sl}}(2)$

$TG(2) = pt \sqcup T^* \mathbb{P}^1 \sqcup pt$

here we now know the stable envelopes

$\Rightarrow$ know the R-matrix $\qquad R = Stab_-^{-1} \circ Stab_+$

Exercise: compute this, and see it is gauge equivalent to Felder's R-matrix

$\Rightarrow$ geometric elliptic $\widehat{\mathfrak{sl}}(2)$ is the same as Felder's

modulo YB, and things $R_{V_1, V_2 \otimes V_3} = R_{V_1,V_2} R_{V_1,V_3}$

easy consequences of general properties of stable envelopes

More generally: $\widehat{\mathfrak{gl}}(n)$ geometrically constructed using Nakajima varieties for the A_{n-1} quiver $\underbrace{\circ - \circ - \circ - \circ}_{n-1.}$

$TG(1) = 2\,\text{pts}$

$TG(2) = pt \sqcup T^* \mathbb{P}^1 \sqcup pt$

$\Longrightarrow$ defining $\mathbb{C}^n$ = pts

$\mathbb{C}^n \otimes \mathbb{C}^n$ = pts $\sqcup$ $T^*\mathbb{P}^1$'s.

in general, $R_{\text{Elliptic}} = R_{\text{Felder}}$ for $\widehat{\mathfrak{gl}}(n)$