July 22, 2020

Wednesday, July 22, 2020 9:57 AM

Enumerative geometry & geometric representation theory

Start time Moscow 17:30
New York 10:30

X = alg. symplectic variety $\circlearrowleft$ torus A, preserves ω_X

$\cap$

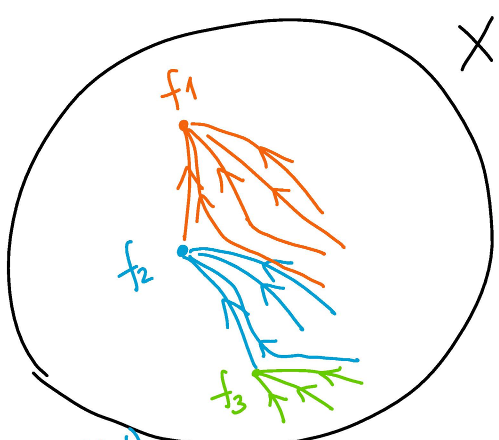
$\circlearrowleft$ larger torus T, scales ω_X with some character $\hbar$

deformation parameter of the quantum group

$$1 \to A \to T \xrightarrow{\hbar} \mathbb{C}^\times \to 1$$

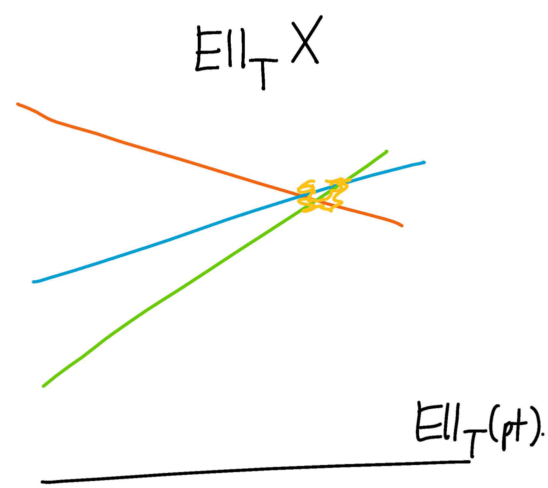
for simplicity, assume $X^A = \{f_i\}$ points

so instead of a Lagrangian in $X \times X^A$, we just need a bunch of Lagrangian subvarieties in X



$Stab(f_i)$ = a section of some line bundle $\mathcal{J}$

we fix the degree of its restriction to fixed pts in the variables a

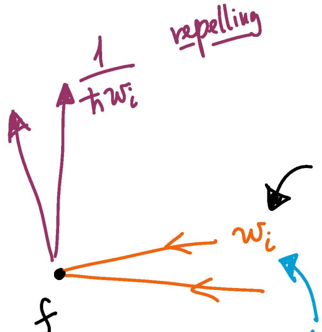


$$\downarrow$$

$Ell_{T/A}(pt)$

coordinate $\hbar$

$\Theta(x) = -\Theta(x^{-1})$



attracting directions

weights of A, $w_i = a^{\mu_i}$

$$Attr(f)\Big|_f = \pm \prod \vartheta(\hbar w_i)$$

has degree $\sum \mu_i^2 \in S^2 \operatorname{char} A$

in the variables A

in particular this is independent of which directions are Attr / Repelling. $\|$ $c_1^2 - 2c_2(-\,''\,-)$

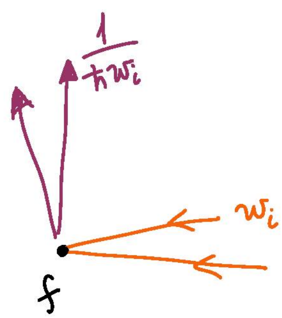
Def. $\mathcal{S}$ is attractive if $\deg_A \mathcal{S}\big|_{\substack{\text{fixed}\\ \text{locus}}} = \deg_A \mathrm{Attr}\big|_{\substack{\text{fixed}\\ \text{locus}}} = \deg_A \Theta(N_{X/\text{fix},<0})$

zero section of $N_{\text{fix},<0}$

repelling

if $\mathcal{S}$ is attractive, $\mathcal{S}\otimes\mathcal{L}$ is attractive for any $\mathcal{L}$ that is of degree 0 or alg. equiv. to 0

a very, very, very important degree of freedom needed for the theory to work!! $\leftarrow$ dynamical variables of elliptic quantum groups



R-matrix $\Leftarrow$ change of Attr/Repelling directions in stable envelopes

$\prod \vartheta(\hbar w_i)$ vs. $\prod \vartheta(w_i)$

$E = \mathrm{Ell}_{\mathbb{C}^*}(pt)$
$\|$
$U(1)$

$$\frac{\vartheta(\hbar w)}{\vartheta(w)\vartheta(\hbar)} = \text{section of Poinare bundle on } \overset{w}{E} \times \overset{\hbar}{E}$$

degree 0, but nontrivial

an element $a \in \mathbb{C}^*$ is like a coordinate on E
$E \ni a \bmod q^{\mathbb{Z}}$.

a section $\Theta(\hbar w - w - \hbar) = \Theta((w-1)(\hbar-1))$

character of a virtual repr.

does not act on X at all

More generally, in $V \in K_T(X)$ take $z \in \mathbb{C}^*$

$\mathcal{U}(V,z) = \Theta\left(\left(V - \mathbb{C}^{\mathrm{rk}V}\right)(z-1)\right) =$ a line bundle on $\mathrm{Ell}_T(X) \times \underset{z}{E}$

of degree 0 in either factor

$\cap$
$\mathrm{Pic}(\mathrm{Ell}_T(X) \times \underset{z}{E})$ we are free/forced to twist by such

$\mathrm{Pic}\left(\mathrm{Ell}_T(X) \times \underset{z}{E}\right)$

we are free / forced to twist by such line bundles in elliptic envelopes

of degree 0 in either factor

<u>Exersize:</u>

$$\mathcal{U}(V, z) = \mathcal{U}(\det V, z)$$

line bundle on X

$$\mathcal{U}(\mathcal{L}, z_1) \otimes \mathcal{U}(\mathcal{L}, z_2) = \mathcal{U}(\mathcal{L}, z_1 z_2)$$

$$\mathcal{U}(\mathcal{L}, z_1 + z_2)$$

Use the theorem of the 

$\mathcal{U}$ is a very general source of line bundles of degree 0

Still need at least one attractive line bundle

polarization of X

<u>Def.</u> $T^{1/2} \in K_A(X)$ such that

$$T^{1/2} + (T^{1/2})^\vee = TX$$

picks half of the tangent vectors

$$\mathcal{J} = \Theta(T^{1/2})$$

has the right degree

$$\Theta(T^{1/2})^{\otimes 2} = \Theta(TX)$$

like the $\sqrt{\ }$ of the canonical

if $X = T^*M \xrightarrow{\pi} M$

then $\pi^* TM$ is a polarization

or $\operatorname{Ker} d\pi$ ——"——

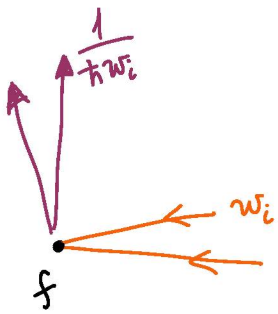
if $\underset{\text{open}}{X \subset} T^*\text{stack}$ same applies

e.g. GIT stable points in $T^*\left[\text{prequotient}/G\right]$

covers Nakajima varieties etc.

$$T^{1/2} = \underbrace{T_{\text{prequotient}} - \mathrm{Ad}(G).}_{G\text{-equivariant, and so descend to } K_T(X)}$$

To reiterate,



$$\mathrm{Attr}\big|_f = \pm \prod_{w_i>0} \vartheta(\hbar w_i)$$

$$\Theta(T^{1/2})\big|_f = \prod_{w_i \in T^{1/2}_f} \vartheta(w_i)$$

index of a fixed pt

$$\left.\frac{\Theta(T^{1/2})}{\Theta(N_{<0})}\right|_f = \mathcal{U}(\det T^{1/2}_{>0}, \hbar)$$

Near the fixed locus: F

$$\text{Stab:} \quad \boxed{\Theta(T^{1/2}_F) \otimes \mathcal{U}(\det \mathrm{ind}, \hbar)^{-1} \longrightarrow \Theta(T^{1/2}_X)}$$

must be there!

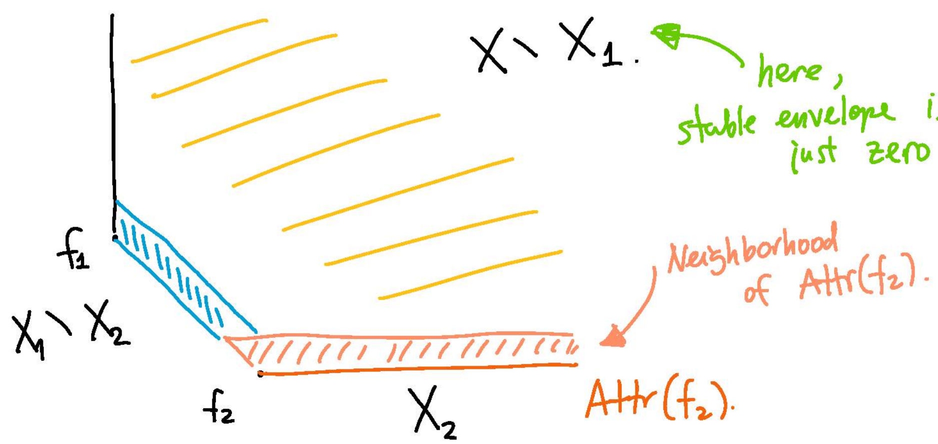
shifts the variables z_i

$$\otimes \bigotimes \mathcal{U}(\mathcal{L}_i, z_i) \qquad\qquad \otimes \bigotimes \mathcal{U}(\mathcal{L}_i, z_i)$$

a basis in Pic(X).

So, finally, we have defined Stab in the neighborhood of fixed locus :)

Extend inductively :



$$X_n = \bigcup_{i \geq n} \mathrm{Attr}(f_i)$$

extend Stab inductively using

$$\longrightarrow H^i(X \setminus X_3, X \setminus X_2) \longrightarrow H^i(X \setminus X_3) \longrightarrow H^i(X \setminus X_2) \longrightarrow \cdots$$

$$\uparrow$$

$$\mathrm{Thom}(N_{\mathrm{Attr}(F_2)})$$

in elliptic cohomology

$$0 \to \Theta^i(-N_{\mathrm{Attr}(F_2)}) \to \mathcal{O}^i_{\mathrm{Ell}_T(X\setminus X_3)} \to \mathcal{O}^i_{\mathrm{Ell}_T(X\setminus X_2)} \xrightarrow{0}$$

$\mathrm{Attr}(F_2) \cong F_2$

$$\Theta^i(-N_{\mathrm{Attr}(F_2)}) = \Theta^i(-N_{X/F_2,<0})$$

repelling

restriction

$\mathcal{O}^i_{\mathrm{Ell}_T(F_2)}$

an embedding

multiplication by $\prod_{w_i<0} \vartheta(w_i)$

all nonzero a-weights

nonzerodivisor !

drop

$$0 \to \Theta^0(-N_{X/F_2,<0}) \to \mathcal{O}_{\mathrm{Ell}_T(X\setminus X_3)} \to \mathcal{O}_{\mathrm{Ell}_T(X\setminus X_2)} \to 0$$

$\otimes \mathcal{J}$

$$0 \to \mathcal{J}|_{F_2} \otimes \Theta(-N_{<0}) \to \mathcal{J}|_{X\setminus X_3} \to \mathcal{J}|_{X\setminus X_2} \to 0$$

$$0 \to H^0(\quad) \to H^0(\quad) \to H^0(\quad) \to$$

¿ interpolation ?

$$\to H^1(\ -\text{"}-\ )$$

should both vanish if line bundle nontrivial

line bundle on has degree 0 on the fibers

$$\begin{array}{ccc} \mathrm{Ell}_T(pt) & \leftarrow & \mathrm{Ell}_A(pt) \\ \downarrow & & \\ \mathrm{Ell}_{T/A}(pt) \ni \hbar & & \end{array}$$

$$\mathrm{Stab}(f_1)\Big|_{X\setminus X_1} = 0$$

extend to $X \setminus X_2$, we carefully set it up so that extend to $X \setminus X_2$, we carefully set it up so that

$$\mathcal{J}\big|_{F_1} \otimes \cdots = \text{trivial}$$

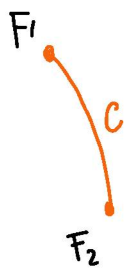
get something nonzero on $X \setminus X_2$, so good

extend to $X \setminus X_3$, need to prove $\mathcal{J}\big|_{F_2} \otimes \cdots =$ is not trivial generically.

<u>Lemma</u> If $\mathcal{L}$ is ample, then

$$\mathrm{wt}_a \, \mathcal{L}\big|_{F_1} \neq \mathrm{wt}_a \, \mathcal{L}\big|_{F_2}$$

so $\otimes\, \mathcal{U}(\mathcal{L}, z)$ <u>will</u> depend on z and so will not be trivial generically



$$0 < \deg \mathcal{L}\big|_C = \frac{\mathrm{wt}\, \mathcal{L}|_{F_1} - \mathrm{wt}\, \mathcal{L}|_{F_2}}{\text{tangent wt at } F_1}$$

If we allow dynamical variables then the family $\mathcal{J}\big|_{F_2} \otimes \cdots$ is nontrivial

map to $\mathrm{Pic}_0(\mathrm{Ell}_A(pt))$, nonconstant.

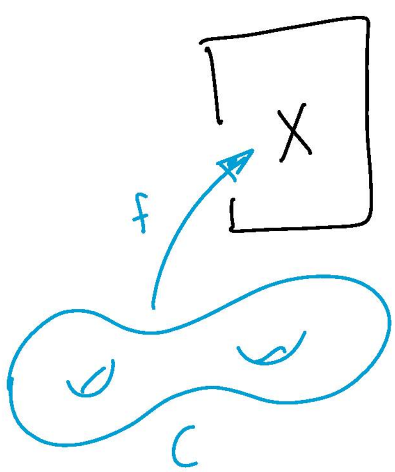
$H^0 = 0$, $H^1 =$ may appear in codim 1.

elliptic stable envelopes are unique

may have poles in z_i and $\hbar$ along divisors where

becomes trivial

through index



$$\sum_d z^d \text{ count of maps of degree } d$$

these are the dynamical variables

$$\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}^* \,/\, q^{\mathbb{Z}}$$

quantum here, classical in elliptic cohomology , poles are a <u>good</u> sign

$T^{1/2}$ is a polar $\iff$ $T^{1/2} + V - V^{\vee}$ is a polar.

$\hbar^{?}$