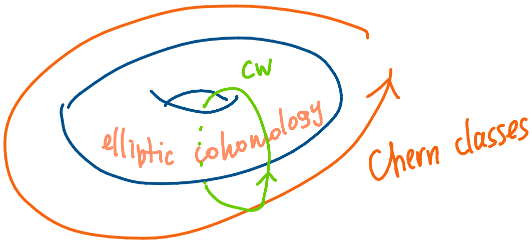
July 8, 2020
Wednesday, July 8, 2020 10:22 AM

Enumerative geometry & geometric representation theory

Start time Moscow 17:30
New York 10:30



$D^b(\mathrm{Coh}\ \mathrm{Spec}\, R)$

$\in$

chain complex

Analogy: $H^\bullet(X,R) = H^\bullet(\ \ \cdots \xrightarrow{d} C^i \xrightarrow{d} C^{i+1} \to \cdots\ )$

ring

R-module, i.e. a sheaf on Spec R

cup product $\cup$: $C^i \otimes_R C^j \to C^{i+j}$

$d(\alpha \cup \beta) = d\alpha \cup \beta + (-1)^{\cdots} \alpha \cup d\beta$

Suspension: $H^{i+1}(\Sigma X) = H^i(X)$, Mayer-Vietoris, ...

reduced

$X \longrightarrow$ complexes of R-modules is triangulated

$C[1]^i = C^{i+1}$ $\leftarrow$ shift by one to the left $\qquad \Sigma = [-1]$

triangle: $A^\bullet \xrightarrow{f} B^\bullet \to \mathrm{Cone}(f) \to A^\bullet[1] \to B^\bullet[1] \to \cdots$

Subcomplex $B^\bullet$ with quotient $A^\bullet[1]$

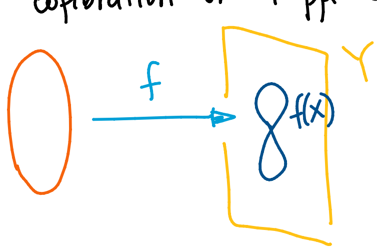
take $A^\bullet[1] \oplus B^\bullet$ with differential $\begin{pmatrix} d_A[1] & 0 \\ f & d_B \end{pmatrix}$

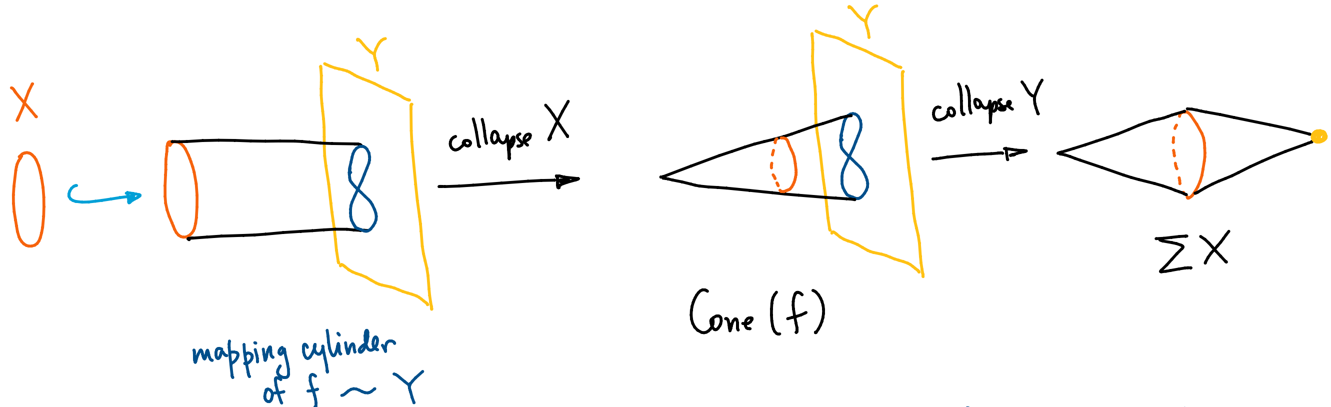
changes sign

because B is sub

For topological space cofibration or Puppe sequence

f: $X =$





$\xrightarrow{\Sigma f} \Sigma Y \to \cdots$

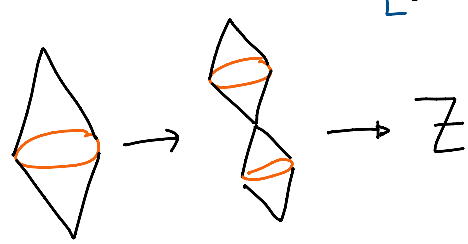
Fundamental fact about this sequence:

$$\leftarrow [\ \ldots\ ] \leftarrow [\Sigma^i X, Z] \leftarrow [\Sigma^i Y, Z]$$

any

is exact

$[\Sigma^1 X, Z]$ is a group

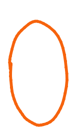


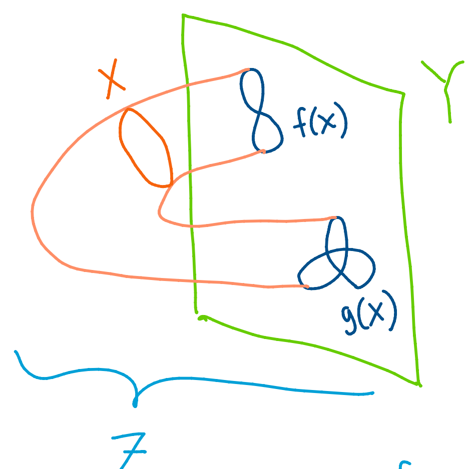
$[\Sigma^{\geq 2} X, Z]$ is an abelian

$H^\bullet$ takes triangles to triangles!

homotopy colimits of spaces $\Longrightarrow$ homotopy limits of sheaves on Spec R

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$





$$H^\bullet(Z) \approx \text{kernel}\left(H^*(Y) \xrightarrow{f-g} H^*(X)\right)$$

$$= \text{Cone}\left(H^*(Y) \xrightarrow{f-g} H^*(X)\right)[-1]$$

Exersize: check that Mayer-Vietoris is a special case of this

<u>Simplest case:</u> glue an n-cell to W along an attachment map f

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d, "f"'] & D^n \arrow[d] \\
W \arrow[r] & W \text{ with this cell glued.}
\end{tikzcd}$$

pushout

In equivariant situation: work with <u>equivariant cells</u> $\quad pt \rightsquigarrow$ orbit of a group G $\{G/H\}$

$D^n \times G/H$

here the group G acts trivially

$$\begin{tikzcd}
S^{n-1} \times G/H \arrow[r] \arrow[d, "f"] & D^n \times G/H \arrow[d] \\
W \arrow[r] & W \cup_f \text{cell}
\end{tikzcd}$$

pushout

$$\mathrm{Cohomology}^\bullet_G(D^n \times G/H) = \mathrm{Cohomology}^\bullet_G(G/H)$$

specified sheaf on $\operatorname{Spec} \mathrm{Cohomology}_G(G/H)$

//

image of structure sheaf under $\operatorname{Spec} \mathrm{Cohom}_H(pt) \to \operatorname{Spec} \mathrm{Cohom}_G(pt)$

compact groups or reductive groups

<u>In equivariant elliptic cohomology</u> have schemes (no need for them to be affine)

abstractly like $\underbrace{}_{}$ G-bundles on $E^\vee$ ← dual elliptic curve.

s.s. of degree 0

GL(1)-bundles on $E^\vee$ degree 0 $= \mathrm{Pic}_0(E^\vee) = E$

Concretely, for connected G

$$\mathrm{Ell}_G(pt) = E \otimes \text{cocharacter lattice}(G) / W$$

Mainly interested in abelian T

$$\mathrm{Ell}_T(pt) = E \otimes \text{cocharacter lattice}(T)$$

$T_1 \to T_2$ induces $\mathrm{Ell}_{T_1}(pt) \to \mathrm{Ell}_{T_2}(pt)$

for instance $\quad T \times T \xrightarrow[\text{product}]{} T \quad$ induces addition on $Ell_T(pt)$

group law

abelian variety.

is a homomorphism for abelian groups

$$1 \longrightarrow \underset{\text{torus}}{\text{quasi}}\ \Gamma \longrightarrow T \longrightarrow T' \longrightarrow 1.$$

$$0 \longrightarrow Ell_P(pt) \to Ell_T(pt) \to Ell_{T'}(pt) \to 0.$$

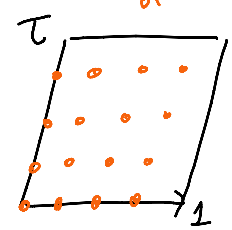
$$T = \mathbb{C}^* \qquad 1 \to \sqrt[n]{1} \to \mathbb{C}^* \xrightarrow{z^n} \mathbb{C}^* \to 1$$

$$\sqrt[n]{1} = \mu_n$$

$$0 \to E[n] \to E \xrightarrow[\text{by } n]{\text{multiplication}} E \to 0$$

points of order n on E

16 pts of order 4



over $\mathbb{C}$ $\quad n^2$ disjoint points

over a field of characteristic $p = n$

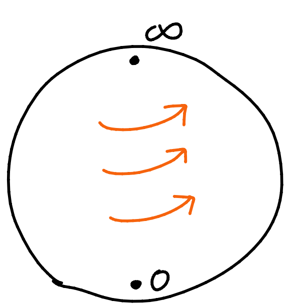
could be p points of fatness p

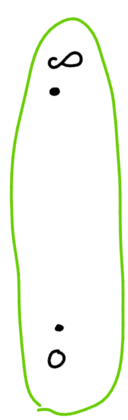
one point of fatness p^2

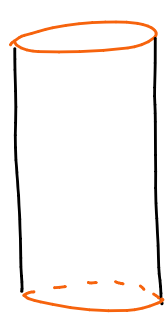
supersingular curves

$$E = \mathbb{C}^*/q^{\mathbb{Z}}$$
$$= \mathbb{G}_m/q^{\mathbb{Z}}$$

Example: $U(1) \hookrightarrow \mathbb{CP}^1$

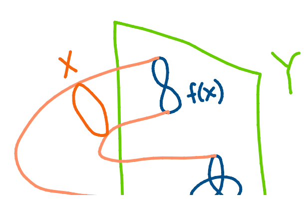






$U(1) \times D^1.$

S^0 $\quad$ 2 orbits $U(1)/U(1)$



This is exactly the situation

2 orbits $U(1)/U(1)$

$Y = 2\text{pts}$ $\qquad X = U(1) = U(1)/\{1\}$

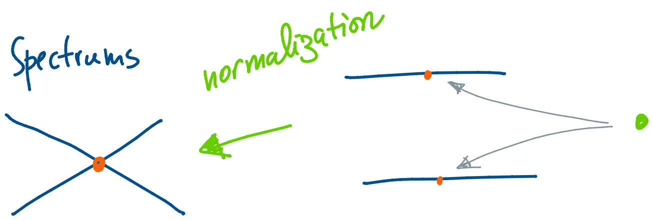
$$0 \to \underset{\varphi_1(0)=\varphi_2(0)}{\text{pairs of functions}} \to \mathcal{O}_E^{\oplus 2} \xrightarrow{\varphi_1(0)-\varphi_2(0)} \mathcal{O}_0 \to 0.$$

(φ_1, φ_2)

because $E = \mathrm{Ell}_{U(1)}(pt)$

$\{0\} \subset E$ is $\mathrm{Ell}_{\{1\}}(pt) \hookrightarrow \mathrm{Ell}_{U(1)}(pt)$.

Take Spectrums

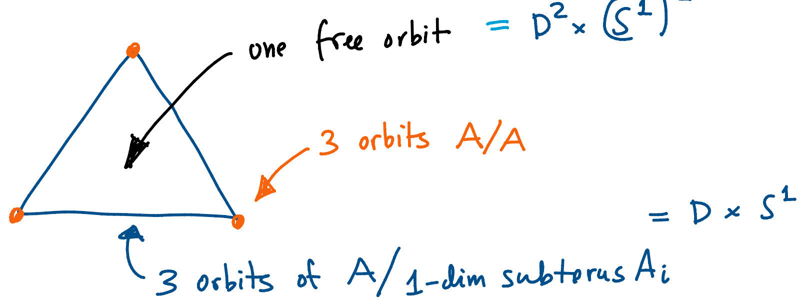


this is what we have computed with chern classes

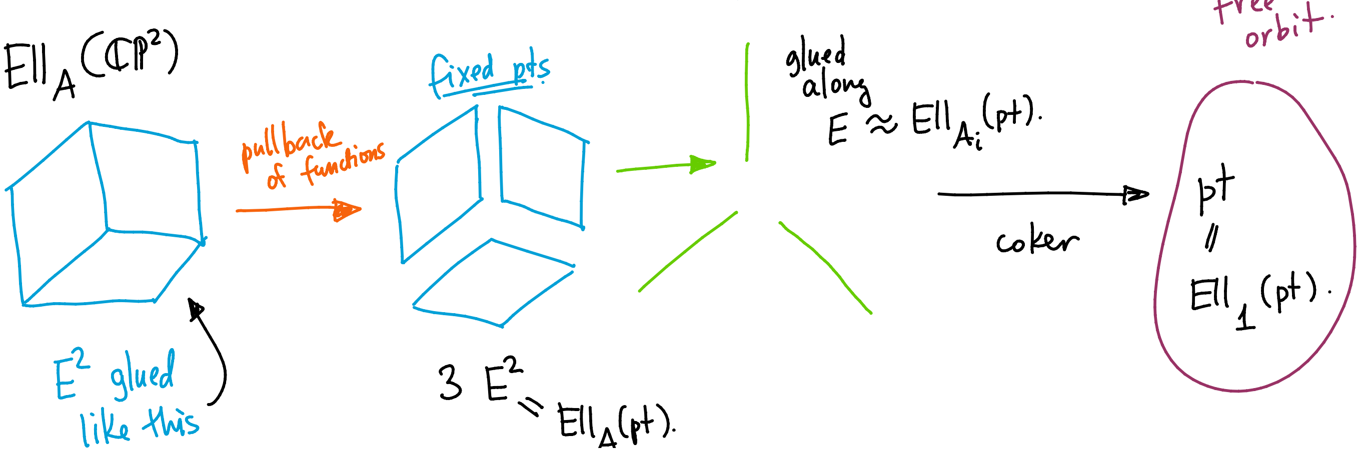
$$0 \to \mathcal{O}_{\mathrm{Ell}_{U(1)}(\mathbb{P}^1)} \to \mathcal{O}_{\text{normalization}} \to \mathcal{O}_{\text{intersection}} \to 0$$

Do the same for $\mathbb{CP}^2 \circlearrowleft$

$$A = \begin{pmatrix} 1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix}$$

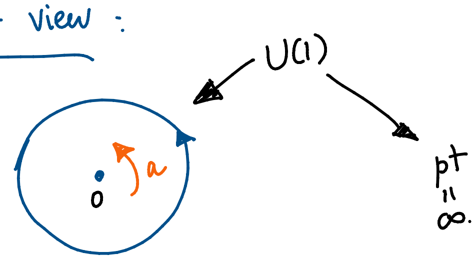


$\mathrm{Ell}_A(\mathbb{CP}^2)$



Different point of view:

$\mathbb{CP}^1 \ni a$



disk bundle in the normal space to 0 in the defining representation a

$$\begin{tikzcd} U(1) \arrow[r] \arrow[d] & B(a) \arrow[d] \\ pt \arrow[r, "\text{pushout}"] & \mathbb{CP}^2 \end{tikzcd}$$

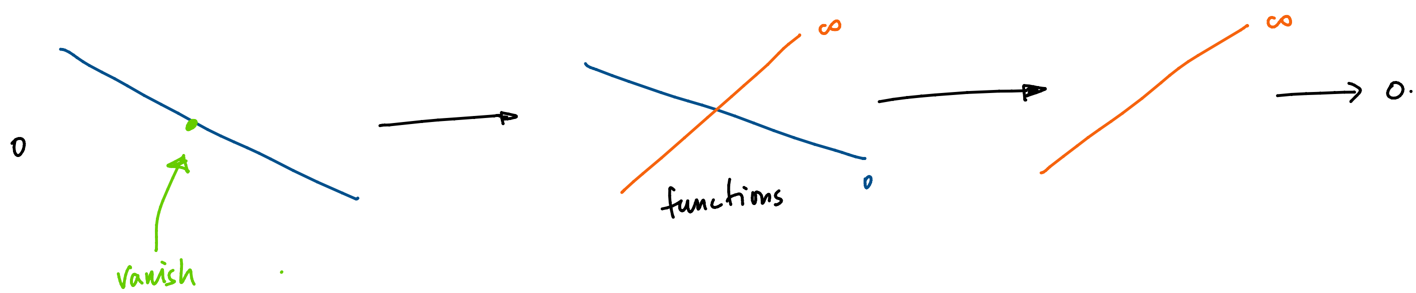
ideal of 0 in $\mathcal{O}_E$

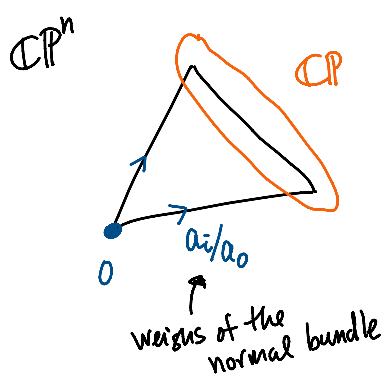
$\mathcal{O}_E$

$\|$

$$0 \to \mathrm{Thom}(a) \to \mathcal{O}_{\mathrm{Ell}_A(\mathbb{CP}^1)} \to \mathcal{O}_{\mathrm{Ell}_A(\infty)} \to 0.$$

v.b. over 0.

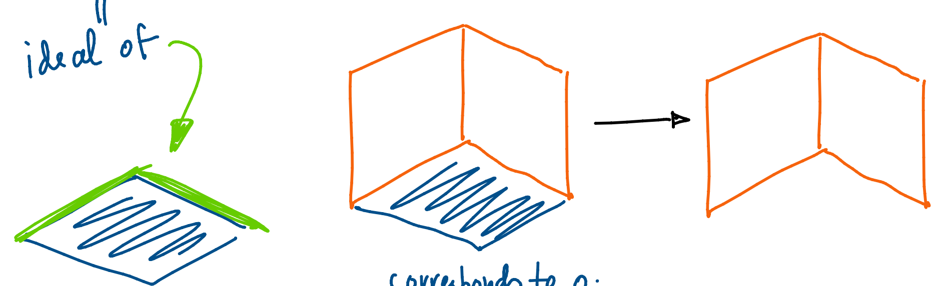




$$0 \to \mathrm{Thom}\left(\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}\right) \to \mathcal{O}_{\mathrm{Ell}(\mathbb{CP}^n)} \to \mathcal{O}_{\mathrm{Ell}(\mathbb{CP}^{n-1})} \to 0$$

$\|$

ideal of



corresponds to 0.

this will be our strategy to construct stable envelopes.