July 1, 2020

Wednesday, July 1, 2020 9:46 AM

Enumerative geometry & geometric representation theory

Start time Moscow 17:30

New York 10:30

$$E = \mathrm{Pic}_0(E^\vee) \quad \text{where} \quad E^\vee = \begin{cases} \text{smooth elliptic curve} \\ \propto \ \text{nodal} \\ < \ \text{cuspidal} \end{cases}$$

$$E = \begin{cases} \text{smooth elliptic curve} \\ \mathbb{G}_m \\ \mathbb{G}_a \end{cases}$$

"spectrum of cohomology"

functor $(G, X) \xrightarrow{\mathrm{Ell}}$ (super) schemes over $\mathrm{Ell}_G(pt)$

compact topological group

(gets more complicated if G is not connected)

G-bundles on $E^\vee$ of $\deg = 0$

reductive

$\oplus$ stables

Jacobi$/\mathbb{C}$ or Tate$/\mathbb{Q}_p$ $\quad E = \mathbb{G}_m / q^{\mathbb{Z}} \quad |q| < 1$

perspective of 2+1 susy QFT

another picture to keep in mind $\quad \mathrm{Ell}(X) \approx K(\mathrm{Loops}(X))$

$$\sum_i (-s)^i \Lambda^i V = 0$$

$$K_G(\mathbb{P}(V)) = K_G(pt)[s^{\pm 1}] \Big/ \prod (1 - v_i s) = 0$$

$GL(V)$

Chern roots of V

functions on $\mathbb{C}^*$ or $\mathbb{G}_m = E$

$\mathrm{Ell}_G(pt)$.

Exactly the same argument applies to

$$\begin{array}{c} G \curvearrowright V \\ \downarrow \\ G \curvearrowright X \end{array}$$

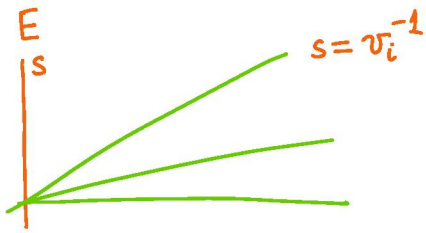
equivariant a vector bundle over X

$$K_{G\times GL(1)}(X) \xrightarrow{\iota_*} K_{G\times GL(1)}(V) \longrightarrow K_G(\mathbb{P}(V)) \longrightarrow 0.$$

$$K_{G\times GL(1)}(X) \ni s$$

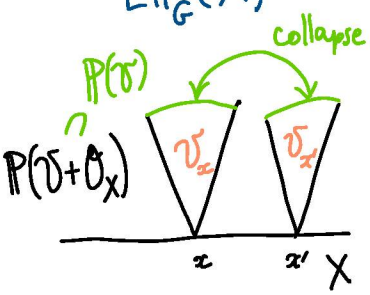
$$K_G(\mathbb{P}(V)) = K_G(X)[s^{\pm 1}] \Big/ \prod(1-sv_i) = 0$$

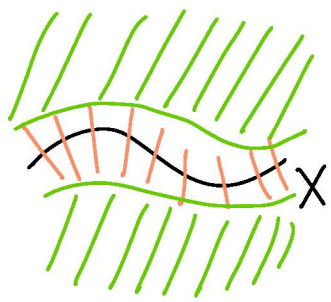
Chern roots



same as before.

$\mathrm{Ell}_G(X)$

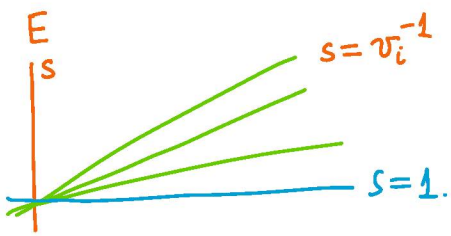




$$\mathrm{Thom}(V) = \mathbb{P}_X(\mathcal{V}+\mathcal{O}_X) \big/ \mathbb{P}_X(\mathcal{V})$$

twisted suspension of X

both in bundle and the G-action.



$\mathrm{Ell}_G(pt).$

$K_{eq}(\mathrm{Thom}(V))$ = functions on  that vanish on 

$\approx$ ideal $\prod(1-v_i)$ in $K_{eq}(X)$

V defines a map $X \longrightarrow pt/GL(r)$

$\uparrow$ of rank r

$$\mathrm{Ell}_{eq}(X) \xrightarrow{c} \mathrm{Ell}_{GL(r)}(pt) = S^r E \longleftarrow S^{r-1}E$$

$(v_1,\dots,v_r) \in S^r E$, $(v_1,\dots,v_{r-1},0) \in S^{r-1}E$

theta divisor D_Θ

bundles with a section.

$$\text{Thom sheaf of } V = c^*(\text{ideal of } \textstyle\prod(1-v_i))$$

$$\text{Thom sheaf of } V = \Theta(-V)$$

$$\text{ideal of } \textstyle\prod(1-v_i) = \mathcal{O}(-D_\Theta).$$

$\Theta(V_1 \oplus V_2) = \Theta(V_1)\otimes\Theta(V_2)$

! Thom isomorphism is <u>not</u> an isomorphism in elliptic cohomology

Speaking of ϑ-functions

$$S^0 E \to S^1 E$$

$$D_\Theta = \{0\} \subset E$$

$$\mathcal{O} \xrightarrow[\vartheta]{} \mathcal{O}(D_\Theta)$$ tautological section

unique

$$1 = \text{origin}$$
$$\cap$$
$$E = \mathbb{C}_m / q^{\mathbb{Z}}$$

$$\vartheta(x) = (x^{1/2} - x^{-1/2}) \prod_{n>0} (1 - q^n x)(1 - q^n/x)$$

$$\vartheta(qx) = -q^{-1/2} x^{-1} \vartheta(x)$$

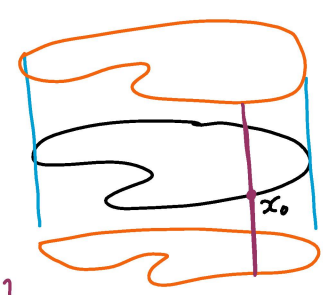
$$\vartheta(x^{-1}) = -\vartheta(x)$$

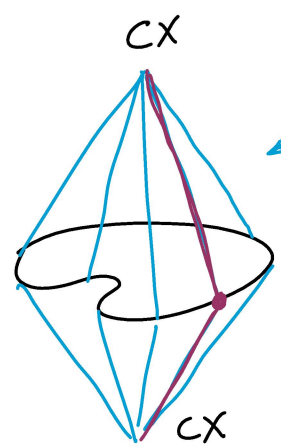
In K-theory $\Theta(-V) \simeq K$, so

$$K_{eq}(\mathrm{Thom}(V)) \cong K_{eq}(X)$$ by mult by $\prod(1 - v_i)$

$$\mathrm{Thom}\left(\begin{matrix}\mathbb{C} \\ \downarrow \\ X\end{matrix}\right) = \Sigma^2 X$$

$$\Sigma^1 X$$
$$\shortparallel$$
$$D^1 \times X / S^0 \times X \cup D^1 \times \{x_0\}$$





vector bundle here is just a clutching function on X

$$K(\Sigma^2 X) = K(X) \qquad \text{Bott periodicity}$$

$\mathbb{C}^r$ bundles over X $= [X, Gr(r,\infty)]$

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ker} \arrow[r] & \mathbb{C}^N \arrow[r] \arrow[dr] & V \arrow[r] \arrow[d] & 0 \\ & & & \bullet\ x & \end{tikzcd}$$

$$= BU(r) = \mathrm{Mat}_{\text{full rank}}(r \times \infty) / GL(r)$$

in K-theory we study bundles up-to stable equivalence that is up-to trivial summands

$$K(X) = [X, \underset{\text{rank}}{\mathbb{Z}} \times BU(\infty)]$$

group up to homotopy w.r.t. $\oplus$.

$$K(\Sigma^1 X) = [X, U(\infty)]$$

$$= [\Sigma^1 X, \mathbb{Z} \times BU(\infty)] = [X, \underbrace{\Omega^1(\mathbb{Z} \times BU(\infty))}_{\simeq\, U(\infty)}]$$

loops

$$U(\infty) \simeq \Omega^1(\mathbb{Z} \times BU(\infty)).$$

$$K(\Sigma^2 X) = [X, U(\infty)]$$

Σ is like $[1]$ for complexes, $K^{-i}(X) = K(\Sigma^i X)$

$$\mathcal{O}^{-2}_{\mathrm{Ell}(X)} = \mathcal{O}_{\mathrm{Ell}(\Sigma^2 X)} = c^* \mathcal{O}(-[0])$$

divisor

$$\begin{tikzcd} E \arrow[r, "c"] & E \arrow[d] \\ & \mathcal{O}(-[0]) \end{tikzcd}$$

zero map

corresponds to a trivial bundle with $c = 0$.

trivial bundle over E with fiber $=$ fiber of $\mathcal{O}(-[0])|_0$

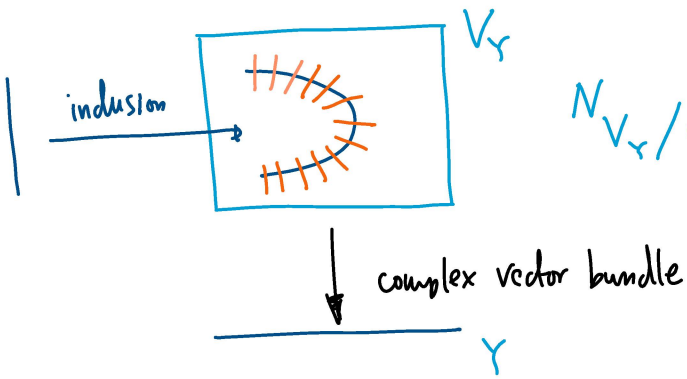
functions that vanis at 0 $= m_0$

$$= m_0 / m_0 \cdot m_0 = T_0^* E$$

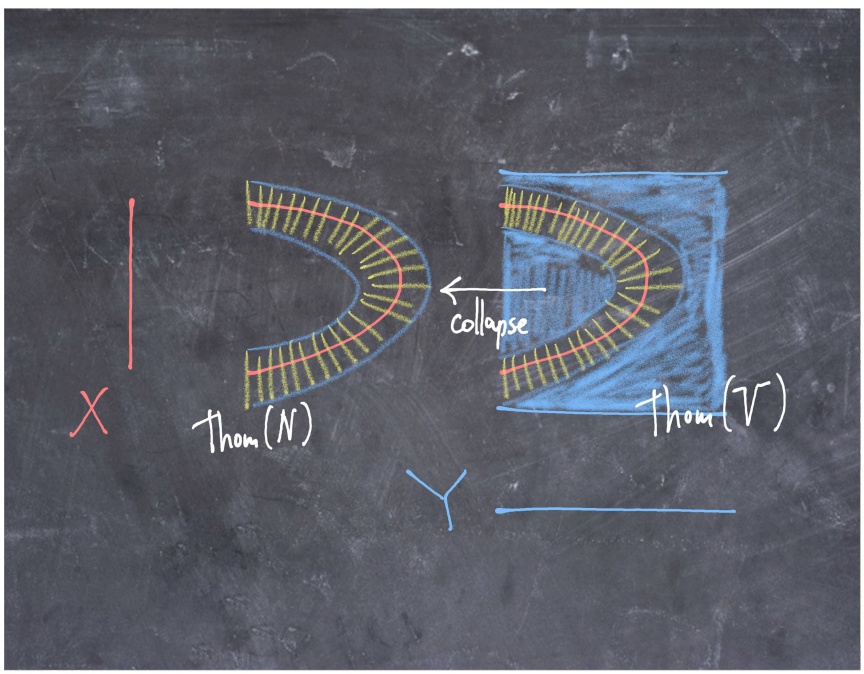
$$\mathcal{O}^{i-2}_{\mathrm{Ell}(X)} = \mathcal{O}^{i}_{\mathrm{Ell}(X)} \otimes \psi_1.$$

line bundle on the moduli of E $= \psi_1$

Pushforward: defined for complex oriented maps $f: X \to Y$



$N_{V_Y / \text{inclusion of } X}$ = also complex.



$$\Theta_X(-N) \simeq \mathcal{O}_{\mathrm{Ell}(\mathrm{Thom}(N))} \longrightarrow \mathcal{O}_{\mathrm{Ell}(\mathrm{Thom}(V))} \simeq \Theta_Y(V).$$

collapse

pushforward: $\Theta_X(-N) \xrightarrow{f_\circledast} \Theta_Y(V)$ map of line bundles

tautological map $\Theta_X(f^*V) \longrightarrow \Theta_Y(V)$

tautological map $\Theta_X(+V) \longrightarrow \Theta_Y(V)$ $\longrightarrow$ $\mathrm{Ell}_{GL(\mathrm{rk}\,V)}(pt)$.

$$f_{\circledast}: \quad \Theta_X(\underbrace{-N_{\text{inclusion}} + f^*V}_{-N_f}) \longrightarrow \mathcal{O}_{\mathrm{Ell}(Y)}$$

projection

$$\boxed{\Theta: K(X) \longrightarrow \mathrm{Pic}(\mathrm{Ell}(X))}$$

<u>Moral:</u> $\int = \dfrac{1}{\text{inclusion}}$

inclusion $\sim \prod \vartheta(x_i)$ Chern roots of the normal bundle

$\int \sim \dfrac{1}{\prod \vartheta(x_i)}$ —"— of the tangent bundle

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One more point of view on computation of $K(\mathbb{P}(V))$

$$\boxed{\mathrm{Ell}_G(G/H \times X) = \mathrm{Ell}_H(X)} \quad \overset{?}{=}$$

in particular $\mathbb{P}^n = \underbrace{U(n+1)}_{G} / \underbrace{U(n) \times U(1)}_{H}$

$$\mathrm{Ell}_G(\mathbb{P}^n) = \mathrm{Ell}_H(pt) = E \otimes_{\mathbb{Z}} \mathrm{cochar}(T)/W_H = E^{n+1}/S(n) =$$
$$= E \times S^n E$$

$$\begin{tikzcd} \mathrm{Ell}_G(\mathbb{P}^n) \arrow[d] & & E \times S^n E \arrow[d] \\ \mathrm{Ell}_G(pt) \arrow[rr, equal] & & S^{n+1}E \end{tikzcd}$$

from $\boxed{}$

$$\begin{tikzcd} \mathrm{Ell}_H(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & \mathrm{Ell}_G(X) \arrow[d] \\ \mathrm{Ell}_H(pt) \arrow[r] & \mathrm{Ell}_G(pt) \end{tikzcd}$$

OK when $G/H = \mathbb{P}^n$ or Gr
$= \mathbb{C}$ or $\mathbb{C}^*$

<u>not true</u> for general G and H
not even in <u>topological K-theory</u>

$= \mathbb{C}$ or $\mathbb{C}^*$

$\mathbb{P}^1 \setminus$ 1pt or 2pts

$\mathrm{Ell}_H(pt) \longrightarrow \mathrm{Ell}_G(pt)$

topological K-theory

$$K_{\underset{\underset{H}{\parallel}}{GL(2)}}(\mathbb{P}^1) \neq K_{\underset{\underset{\underset{\underset{G}{\parallel}}{PGL(2)}}{\parallel}}{Aut}}(\mathbb{P}^1) \otimes_{K_{Aut}(pt)} K_H(pt).$$

not surjective, $\mathcal{O}(1)$ is not in the image.