$$E = \text{Pic}_0(E^\vee)$$ where \( E^\vee = \{ \begin{array}{l} \text{smooth elliptic curve} \\
\text{nodal} \\
\text{cuspidal} \end{array} \) 

\[ E = \begin{cases} 
\text{smooth elliptic curve} \\
G_m \\
G_a 
\end{cases} \]

functor \((G, X) \rightarrow \text{Ell} \) (super) schemes over \( \text{Ell}_G(\text{pt}) \)

compact topological group

(gets more complicated if \( G \) is not connected)

Jacobi \( / \mathbb{C} \) or Tate \( / \mathbb{Q}_p \) \( E = G_m / q \mathbb{Z} \ | q \mathbb{Z} \)

another picture to keep in mind \( \text{Ell}(X) \approx K(\text{Loops}(X)) \)

\[ K_G(\text{IP}(V)) = K_G(\text{pt}) \left[ s^{-1} \right] / \prod (1 - \psi_i s) = 0 \]

\[ \sum (-s)_i \wedge i \mathcal{V} = 0 \]

chern roots of \( V \)

functions on \( G \times \mathbb{G}_m \Rightarrow E \)

\( \mathbb{G}_a \Rightarrow \mathbb{E} \)

Exactly the same argument applies to \( G \rightarrow V \) \( G \times \mathcal{X} \)

equivariant a vector bundle over \( X \)
Theorem (V) = \frac{\mathbb{P}_X (\mathcal{O} + \mathcal{O}_X)}{\mathbb{P}_X (\mathcal{O})}

twisted suspension of X

\text{Thom} (V) = \text{functions on } \text{that vanish on } \mathbb{P}_X (\mathcal{O} + \mathcal{O}_X) / \mathbb{P}_X (\mathcal{O})

\text{Thom Isomorphism is not an isomorphism in elliptic cohomology}
Speaking of $\Theta$-functions

1 = origin

$E = \mathbb{G}_m/\mathbb{Q}$

\[
\Theta(x) = (x^{\frac{1}{n}} - x^{-\frac{1}{n}}) \prod_{n>0} (1-q^n x)(1-q^n/x)
\]

\[
\Theta(qx) = -q^{-\frac{1}{2}} x^{-\frac{1}{2}} \Theta(x)
\]

\[
\Theta(x^{-1}) = -\Theta(x)
\]

In K-theory $\Theta(-V) \subset K$, so

$K_{eq}(\text{Thom}(V)) \cong K_{eq}(X)$ by mult by $\prod(1-v_i)$

$\text{Thom}(\mathbb{C}) \downarrow x = \Sigma^2 X$

\[
\Sigma^1 X
\]

$D^4 \times X / S^0 \times X \cup D^4_x \{x_0\}$

Vector bundle here is just a clutching function on $X$
\[ K(\Sigma^2 X) = K(X) \quad \text{Bott periodicity} \]

C^r bundles over \( X \) = \([X, \text{Gr}(r, \infty)]\)

\[ \operatorname{ker} \to C^N \to V \to 0 \]

\[ X \]

\[ \text{BUn}(r) = \text{Mat}_{\text{full rank}}(r \times \infty) / \text{GL}(r) \]

in \( K \)-theory we study bundles up to stable equivalence that is up to trivial summands

\[ K(X) = \left[ X, \mathbb{Z} \times \text{BU}(\infty) \right] \]

\[ K(\Sigma^{1} X) = \left[ X, U(\infty) \right] \]

\[ = \left[ \Sigma^{1} X, \mathbb{Z} \times \text{BU}(\infty) \right] = \left[ X, \Omega^{1} \left( \mathbb{Z} \times \text{BU}(\infty) \right) \right] \]

\[ \text{loops} \]

\[ U(\infty) \]

\[ \Omega^{1}(\mathbb{Z} \times \text{BU}(\infty)) \]

\[ \Sigma \text{ is like } [1] \text{ for complexes} \quad K^{-i}(X) = K(\Sigma^{i} X) \]

\[ O^{-2}_E \text{ Ell}(X) = O^{-2}_E \text{ Ell}(\Sigma^{2} X) = c^* O(-[0]) \]

corresponds to a trivial bundle with \( c = 0 \).

trivial bundle over \( E \) with fiber = fiber of \( O(-[0]) \) \( \big|_0 \)

\[ O^{-2}_E \text{ Ell}(X) = O^{i} \text{ Ell}(X) \otimes \psi_{\lambda} \]

functions that vanishes at 0

\[ m_0/m_0 m_0 = T^*E \]
Pushforward: defined for complex oriented maps $f: X \to Y$

Line bundle on the moduli of $E = \Psi_*$

$N_{\mathbb{V}_Y} / \text{inclusion of } X = \text{also complex.}$

Complex vector bundle

$\mathbb{H}_X(-N) \simeq \mathcal{O}_{\text{Ell}(\text{Thom}(N))} \xrightarrow{\text{collapse}} \mathcal{O}_{\text{Ell}(\text{Thom}(V))} \simeq \mathcal{O}_Y(V).$

Pushforward: $\mathbb{H}_X(-N) \xrightarrow{\text{f*}} \mathbb{H}_Y(V)$ map of line bundles

Tautological map $\mathbb{H}_X(f^*V) \xrightarrow{} \mathbb{H}_Y(V)$
tautological map \( \Omega_X(\pm V) \rightarrow \Omega_Y(\nu) \)

\[ f : \Omega_X( - \text{inclusion} + f^*V ) \rightarrow \mathcal{O}_{\text{Ell}(Y)} \]

projection \(-N_f\)

\[ g : K(X) \rightarrow \text{Pic}(\text{Ell}(X)) \]

**Moral:** \( S = \frac{1}{\text{inclusion}} \)

\( \text{inclusion} \sim \sum_j \Theta(x_j) \)

\( S \sim \frac{1}{\sum_j \Theta(x_j)} \)

Chern roots of the normal bundle

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One more point of view on computation of \( K(P(V)) \)
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\[ \text{Ell}_G(\text{G/H} \times X) = \text{Ell}_H(X) \]

in particular \( P^n = U(n+1)/U(n) \times U(1)_G \times U(1)_H \)

\[ \text{Ell}_G(P^n) = \text{Ell}_H(\text{pt}) = E \otimes \text{cochar}(T)/W_H = E^n / S(n) = E \times S^n E \]

\[ S^n E \]

From \( G/H = P^n \) or \( G \neq C^* \)

\( \text{Ell}_H(X) \rightarrow \text{Ell}_G(X) \)

not true for general \( G \) and \( H \)

not even in topological \( K \)-theory
\[ K_{GL(2)}(\mathbb{P}^1) \neq K_{\text{Aut}}(\mathbb{P}^1) \otimes K_{\text{Aut}(\pt)} K_H(\pt). \]

Not surjective, \( \Theta(i) \) is not in the image.