June 24, 2020

Tuesday, June 23, 2020 11:19 PM

Enumerative geometry & geometric representation theory

Start time Moscow 17:30

New York 10:30

$$K_{GL(V)}(\mathbb{P}(V)) = ? \qquad \mathbb{P}(V) = (V \setminus 0)/GL(1)$$

$$K_G(Y) = K(Y/G)$$

Ex. $K_G(G/H \times X) = K_H(X)$

$$\| $$

$$K_{\underbrace{GL(V)\times GL(1)}_{G}}(V\setminus 0)$$

Consider $\iota: \{0\} \to V \xrightarrow{\pi} \{0\}$

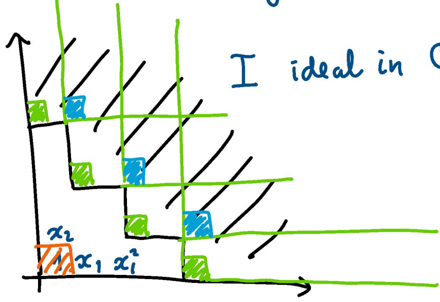
$$K_G^1(V\setminus 0) \to K_G(pt) \xrightarrow{\iota_*} K_G(V) \xrightarrow{\text{restr.}} K_G(V\setminus 0) \to 0$$

$K_G(pt) \cong R(G)$, $\xrightarrow{\pi^*}$ is an isomorphism, $K_G(V) \cong R(G)$

need to compute equivariant resolution of $\mathcal{O}_0 \in K_G(V)$

$G \supset$ maximal torus T = diagonal matrices

$\dim V = 2$

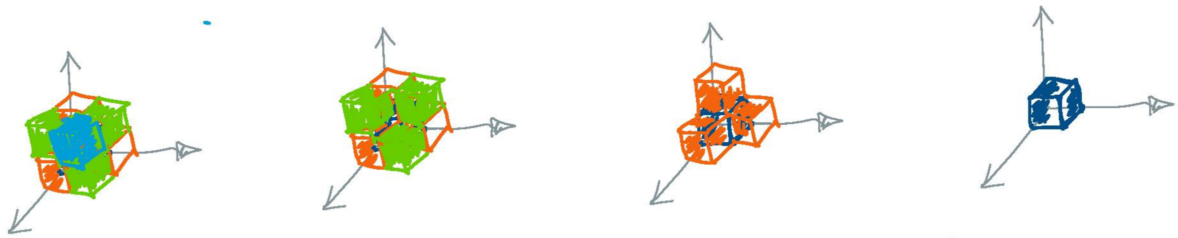


I ideal in $\mathbb{C}[x_1,x_2]$

$$0 \to \bigoplus R\,\blacksquare \to \bigoplus R\,\blacksquare \to R\,\blacksquare \to \underset{\underset{R}{\|}}{\mathbb{C}[x_1,x_2]/I} \to 0.$$

$\dim V = 3$

$$0 \to R x_1x_2x_3 \to \bigoplus_{i<j} R x_i x_j \to \bigoplus R x_i \to R \to \mathcal{O}_0 = R/(x_1,x_2,x_3)$$



$$0 \to \underset{s^{-3}}{\Lambda^3 V^* \otimes \mathcal{O}_V} \xrightarrow{d} \underset{s^{-2}}{\Lambda^2 V^* \otimes \mathcal{O}_V} \xrightarrow{d} \underset{s^{-1}}{V^* \otimes \mathcal{O}_V} \xrightarrow{d} \mathcal{O}_V \to$$

$$d = \sum_i \frac{d}{dx_i} \otimes x_i$$

in the odd sense, anticommute $\left(\frac{d}{dx_i}\right)^2 = 0$, $\frac{d}{dx_i}\frac{d}{dx_j} = - \ldots$

$s \in GL(1)$ acts by s^{-k} on $\Lambda^k V^*$

upon $K_G(V\setminus 0) = K_{G/GL(1)}(\mathbb{P}(V))$

$s \longrightarrow \mathcal{O}(1)$

$$K_{GL(V)}(\mathbb{P}(V)) = K_{GL(V)}(pt)\,[s^{\pm 1}] \Big/ \sum (-s)^i \Lambda^i V = 0$$

<u>In coordinates:</u>

representation ring, i.e.

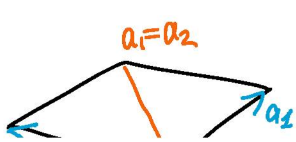
polynomials in $\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$, symmetric in the a_i, (W- invariant)

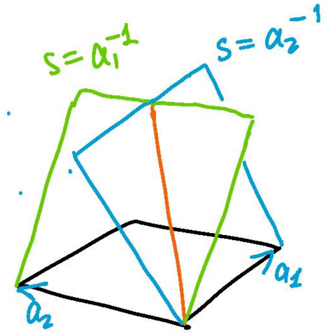
$$\Lambda^3 V = a_1a_2a_3 + a_1a_2a_4 + \cdots = e_3(a_1, a_2, \ldots)$$

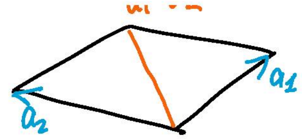
$$K_{GL(V)}(\mathbb{P}(V)) = \text{symm poly in } a_i \,[s^{\pm 1}] \Big/ \prod (1 - s a_i) = 0$$

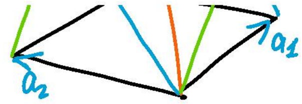
$$= \bigcup_i \{s = a_i^{-1}\}$$

$n=2$

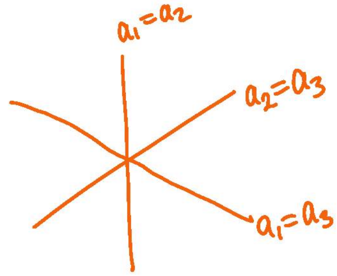


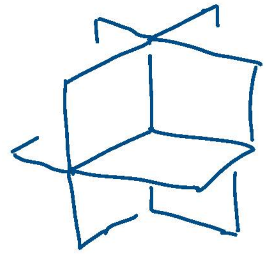






$n=3$





Lessons ①

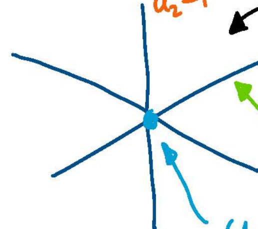
$$G = \begin{cases} G_{\text{compact}} \\ \uparrow\downarrow \\ G_{\text{alg}} = \text{split reductive group} = \operatorname{Spec} \bigoplus_{\substack{\text{irreps } V \\ \text{of } G_{\text{compact}}}} \operatorname{End}(V) \\ \cup \\ G_{ss} \text{ semisimple} \end{cases}$$

$$\begin{array}{ccc} \operatorname{Spec} K_G(X) & \longleftarrow & K(X^g) \\ \downarrow & & \downarrow \\ \operatorname{Spec} K_G(pt) = G_{ss}/G & \longleftrightarrow & \{g\} \\ \| & & \\ T/W & & \end{array}$$

stratification by types of fixed loci

$\prod(1-s a_i)=0.$

$a_1=1$



$(1-s)(1-a_2 s)(1-a_3 s)$ 3pt $= K(\mathbb{P}^2{}^{\text{generic}})$.

$(1-s)(1-a_2 s)^2 = K\left(\mathbb{P}^2{}^{\begin{pmatrix}1 & & \\ & a_2 & \\ & & a_2\end{pmatrix}}\right)$ $= pt \sqcup \mathbb{P}^1$

$(1-s)^3=0$ 1pt of length 3 $= K(\mathbb{P}^2)$.

"Localization" over generic pt we have $K(X^T)$, $T\subset G$ maximal torus

In elliptic cohomology: G/G = G-local system on  $= \operatorname{Spec} K_G(pt)$.

monodromy, up to conjugation

$\operatorname{Ell}_{G_c}(pt)/\mathbb{C}$ = G_c-local systems on 

scheme

$\| \; NS, \; g=1$

complex elliptic curve

$\bigoplus$ Stable G-bundles on $E^\vee$ of degree 0 $\quad G=GL(n)$

$\operatorname{Pic}_0(E^\vee)=E$

$\| \; g=1$

$\bigoplus$ line bundles of degree 0 / permutation

$= E^{\operatorname{rk} G}/W$

$$\mathrm{Ell}_G(pt) = E \otimes_{\mathbb{Z}} \mathrm{cochar}(T) / W$$

$\mathrm{Ell}_H(pt) \to \mathrm{Ell}_G(pt)$ $\qquad$ $\mathrm{Ell}_{U(1)}(pt) = E$

$$\mathrm{Ell}_{GL(V)}(\mathbb{P}(V)) \subset \underset{a_1, \dots, a_n}{E^n / S(n)} \times \underset{s}{E} \quad \text{is} \quad \bigcup_i \{ s a_i = 1 \}$$

$\dim V = n$

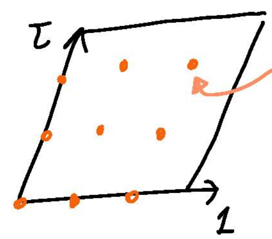
$\mathrm{Ell}_G(X)$ is covariant

w.r.t. $(G_1, X_1) \to (G_2, X_2)$

<u>Ex.</u> $1 \to \mu_3 \to GL(1) \xrightarrow{z^3} GL(1) \to 1$

$$0 \to E[3] \to E \xrightarrow{3} E \to 0$$

$$E = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$$



pairs of commuting elements of μ_3

② <u>Chern classes.</u> $\mathcal{P}$ = principal H bundle, G-equivariant

$$\mathcal{P} \downarrow \quad G \curvearrowright X \qquad \longleftrightarrow \quad X \to [pt/H]$$

$$\mathrm{Spec}\, K_G(X) \to \mathrm{Spec}\, K_{G \times H}(pt).$$

chern class

Concretely, rank n vector bundle V on X gives

$$\mathrm{Spec}\, K_G(X) \to K_{GL(n)}(pt) = \text{symmetric polynomials in } x_1, \dots x_n$$

"Chern roots"

$$\Lambda^k V \xleftarrow{\text{pull-back}} e_k(x_1, \dots, x_n)$$

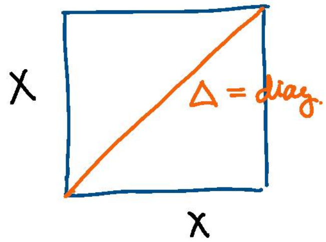
Literally the same for elliptic cohomology $\mathrm{Ell}_{GL(n)}(pt) = E^n / S(n) = S^n E$

Some collection of vector bundles $V_1, V_2, \dots$ generate $K_G(X)$ as λ-ring (or with Adams operations, ..)

take $\Lambda^k V_i$

$$\mathrm{Spec}\, K_G(X) \hookrightarrow K_G(pt) \times \prod S^{\mathrm{rk}\, V_i} E$$

is an embedding. E.g. $\mathcal{O}(1)$ generates $K(\mathbb{P}^n)$ spans K.



$$\mathcal{O}_\Delta = \sum \mathcal{F}_i \boxtimes \mathcal{G}_j$$

in the algebra generated by V_i

for instance on $\mathbb{P}^n$ we have $0 \to \underset{\text{sub}}{\mathcal{O}(-1)} \to \mathbb{C}^{n+1} \to \text{quotient} \to 0.$

on $\mathbb{P}^n \times \mathbb{P}^n$ considen $\text{sub}_1 \to \mathbb{C}^{n+1} \to \text{quotient}_2.$

$$\text{sub}_1 \xrightarrow{s} \text{quotient}_2$$

$s = \text{Hom}(\text{sub}_1, \text{quotient}_2)$

$= \mathcal{O}(1) \boxtimes \text{quotient}$

$\Delta = \{ s = 0 \}$ $\quad$ Koszul(s) $\overset{\shortparallel}{}$ resolution of $\mathcal{O}_\Delta$

on $\text{Hilb}(\mathbb{C}^2, n)$ we have $0 \to I \to \mathbb{C}[x_1, x_2] \to \underset{\mathbb{C}[x_1,x_2]/I}{\text{quotiend}} \to 0$

rank n.

generates K-theory.

$$\text{Spec}\, K_T \text{Hilb}(\mathbb{C}^2, n) \subset T \times \overbrace{K_{GL(n)}(pt)}^{\text{Spec}}$$

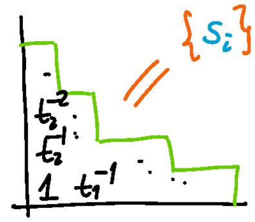
$T = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \subset GL(2) \circlearrowright \mathbb{C}$

acts on $x_1, x_2 \in (\mathbb{C}^2)^*$

$\begin{pmatrix} t_1^{-1} & 0 \\ 0 & t_2^{-1} \end{pmatrix}$

is a union over all fixed pts

$s_1, \dots, s_n / S(n)$



$= \{s_i\}$ $\quad |\lambda| = n$

$\text{Ell}_T \text{Hilb}(\mathbb{C}^2, n) = \text{same}$

(3) $\quad \mathcal{E}$ vector bundle
$\downarrow$
X

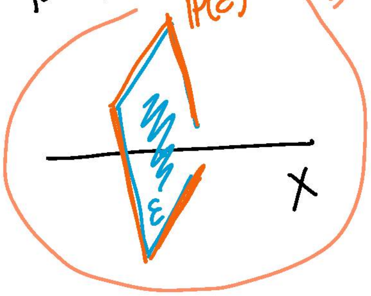
literally same

$$K_G(\mathbb{P}(\mathcal{E})) = K_G(X)[s^{\pm 1}] \Big/ \sum_i (-s)^i \Lambda^i \mathcal{E} = 0.$$

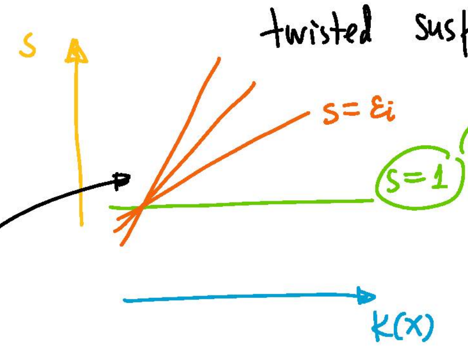
for instance $\mathbb{P}(\mathcal{E}) \subset \mathbb{P}_X(\mathbb{C} + \mathcal{E})$

$$\text{Thom}(\mathcal{E}) = \mathbb{P}_X(\mathbb{C} + \mathcal{E}) \Big/ \mathbb{P}_X(\mathcal{E})$$

for instance $\mathbb{P}(\mathcal{E}) = \mathbb{P}_X(\mathbb{C}+\mathcal{E})$ $\qquad$ $\mathrm{Thom}(\mathcal{E}) = \mathbb{P}_X(\mathbb{C}+\mathcal{E})/\mathbb{P}_X(\mathcal{E})$



twisted suspension of X



ε_i = chern roots of $\mathcal{E}$

functions on the whole thing that vanish on the red part.

= ideal generated by $\prod(1-\varepsilon_i)$ in $K(X)$

as a sheaf over $K(X)$ is $\simeq K(X)$ $\qquad$ Thom isomorphism

not so for <u>elliptic cohomology</u>