# G-bordisms. Lecture

Wednesday, June 17, 2020 4:35 PM

- Cohomology
- K-theory
- Elliptic cohomology
- Unitary cobordisms

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## (Extraodinary) Cohomology

$h^*(X, A)$ is a functor from category of pair of topological spaces (CW complexes) to abelian category

1. functor
2. homotopy
3. long exact sequence
4. excision
5. disjoint union
6. ~~$H^n(*) = 0 \quad n > 0$~~

"Nice" theory

Complex vector bundles are $h^*$-orientable

Tom isomorphism

$$\begin{array}{c} \xi \\ \downarrow \\ X \end{array} \text{ vector bundle}$$

$$h^*(X) \cong h^*(B(\xi), S(\xi))$$

$$t(\xi) \in h^{2n}(B, S) \qquad i : X \to B$$

$$i^* t = e(\xi)$$

Gizin exact sequence

$$\to h^{*-2n}(X) \xrightarrow{e(\xi)} h^*(X) \to h^*(S) \xrightarrow{\delta} h^{*-2n+1}(X)$$

$$h^*(\mathbb{C}P^\infty) = h^*[[u]] \qquad u = e(\eta) \text{ canonical line bundle}$$

Brown representation theorem

Spectrum $\quad M_n \to M_{n+1} \to \cdot$

$$S^1 \wedge M_n \to M_{n+1}$$

$$h^*(X,A) = \operatorname{dir} \lim_{k\to\infty} [S^k \wedge X/A, M_{n+k}]$$

$$h_*(X,A) = \operatorname{dir} \lim_{k\to\infty} [S^{k+n}, X/A \wedge M_k]$$

1-dim Formal (commutative) group

$$f(u,v) \in K[[u,v]] \qquad K\text{-ring}$$

1. $f(u,0) = u$
2. $f(u,v) = f(v,u)$
3. $f(f(u,v),w) = f(u, f(v,w))$

$$f(u,v) = u+v + \sum_{i+j>1} \alpha_{ij} u^i v^j \qquad \alpha_{ij} \in K$$

- formal group of cohomology theory

$$f_h(u,v) = c_1(\xi \otimes \eta) \in h^*(\underset{\xi}{\mathbb{C}P^\infty} \times \underset{\eta}{\mathbb{C}P^\infty})$$

<u>Ex</u>

1. $f(u,v) = u+v$ additive
2. $f(u,v) = u+v+uv$ multipliaative

$$3 \quad f(u,v) = \frac{u\sqrt{P(v)} + v\sqrt{P(u)}}{1-\varepsilon u^2 v^2} \quad \text{elliptic}$$

$$P = 1 - 2\delta t^2 + \varepsilon t^4$$

$$K = \mathbb{Z}[\tfrac{1}{2}][\delta, \varepsilon]$$

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Lazard: $\exists \quad F(u,v) \qquad \hat{K}$

s.t. for $f(u,v)$ $\exists$ $\tilde{f} : \hat{K} \to K$

$$f(u,v) = \tilde{f}(F(u,v))$$

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Over $\mathbb{Q}$ all formal groups are isomorphic

$$f(u,v) = g^{-1}(g(u)+g(v))$$

$$g = u + \sum \lambda_i u^i$$

$$g(u) = \sum_{n=0}^{\infty} \frac{x_n}{n+1} u^{n+1} \qquad x_n \text{ - independent variables}$$

Quillen (Mischenko, Novikov, Buchstaber)

formal group of unitary cobordism is Lazard's formal group

$$g(u) = \sum_{i=0}^{\infty} \frac{[\mathbb{C}P^n]}{n+1} u^{n+1}$$

$$h^*(\quad) \to \underline{f(u,v)}$$

$$\underbrace{\tilde{h} : U^*(pt) \to h^*(pt),}$$

• $\underline{h^*(X,Y) \cong U^*(X,Y) \otimes_{\hat{h}} h^*(pt)}$ as $\mathbb{Z}$-graded theories

$\tilde{h}$ is Landweber flat

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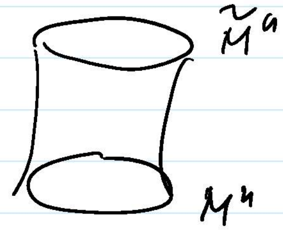
## Unitary bordisms

$$\Omega(X,A) = [M, \partial M \to X, A] / \sim \text{ bord equivalence}$$




$$M^n \sim \tilde{M}^n$$

$$\partial W^{n+1} = M^n \cup (-\tilde{M}^n)$$

$\underline{U_*(X,A)}$ unitary bordisms are bordisms with complex structure in a stable tangent bundle

$U^*(X,A)$ Milnor-Novikov

Spectrum $MU(n) \to$

$$U_{2n}(pt) \cong U^{-2n}(pt)$$

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## Hirzebruch multiplicative genera

$$\hat{h} : U^*(pt) \cong U_*(pt) \xrightarrow{\text{homomorphis}} Q$$

$$\underline{h(x) = x + \sum \lambda_i x^i}$$

$$\hat{h} : U^*(pt) \to Q$$

$$h(X) = < \prod^n \frac{x_i}{\ldots}, X >

$$\underbrace{h(X) = \left\langle \prod_{i=1}^{n} \frac{x_i}{h(x_i)}, X \right\rangle}$$

$$\prod (1 + x_i) = 1 + c_1 + c_2 \cdots$$

$$g_h(u) = \sum \frac{h([CP^n])}{n+1} u^{n+1}$$

$$h(x) = g_h^{-1}(x) \qquad \text{Novikov's observation}$$

Todd genus $\quad Td[CP^n] = 1$

Sign $\quad Sign[CP^{2n}] = 0 \quad Sign(CP^{2n+1}) = 0$

A-genus for Spin manifold

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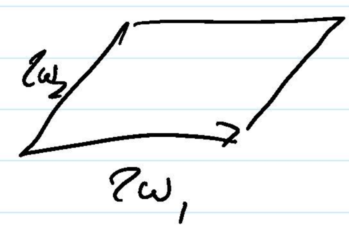
Generalized elliptic genus $\quad$ 4 parametric family of formal groups

$$\boxed{\Phi(x, z \mid \omega_1, \omega_2) = \frac{\sigma(z-x)}{\sigma(x)\sigma(z)} e^{\zeta(z) x}}$$

$$\Phi(x, z + 2\omega_\alpha \mid \quad) = \Phi(x, z \mid \quad)$$




$z = 0$ essential singularity

$$\Phi(x + 2\omega_\alpha, z \mid \quad) = \Phi(x, z \mid \quad) e^{(2\zeta(z)\omega_\alpha - 2\eta_\alpha z)}$$

$$\Phi(x, z \mid \quad) = \frac{1}{x} + O(1) \qquad \text{near } x = 0$$

$$\hat{\varphi}(x, z, k \mid \omega_1, \omega_2) = \frac{e^{kx}}{\Phi(x, z \mid \omega_1, \omega_2)}$$

<u>Claim</u> All classical genera + elliptic one are particular casses of this generalized elliptic genus

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<u>Lame equation</u> $$\left(\frac{d^2}{dx^2} - 2\wp(x)\right)\Phi(x,z) = \wp(z)\Phi(x,z)$$

Addition theorem

$$\Phi(x+y)\left[\wp(y) - \wp(x)\right] = \Phi'(x)\Phi(y) - \Phi(x)\Phi'(y)$$

$$\Phi(x,z)\ \Phi(-x,z) = \wp(z) - \wp(x) \qquad \forall z$$

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$\Phi(x, \omega_\alpha | \omega_1, \omega_2)$ – one gets elliptic formal

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<u>Equivariant bordism theory</u>

<u>Def</u> $$U_*^G(X) = \underset{\text{equivariant}}{[M, X]} / \sim \qquad G \circlearrowright X$$

$U_G^*(X)$ ?

If $X$ is a free $G$-manifold (space)

$$U_G^*(X) = U^*(X/G)$$

Let $X$ be $G$-space

$U_G$

$$X_G = (X \times EG)/G \qquad \text{Borel construction}$$

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$$\downarrow X$$
$$BG$$

$$\frac{U^*(X_G)}{\chi \in U^*(BU)}$$

Equivariam characteristic class

If $\xi$ is $G$ vector bundle over $X$ then by Borel construction we

$$\begin{array}{ccc} \xi_G & = & \xi \times EG/G \\ \downarrow & & \\ X_G & = & X \times EG/G \end{array}$$

$$\chi^G(\xi) := P_! \; \underline{\chi(\xi_G)} \in U^*(BG)$$

Localization theorem

Let $H \subset G$ be a normal subgroup of $G$ and Let $\underline{\underline{F_S}}$ be connected components of fixed paints set of $H$ action

$$\boxed{p_*\left(\chi(\xi_G)\right) = \sum_s p_{s*}\left(\frac{\chi(\xi_{sG})}{e(\nu_{sG})}\right)}$$

Equivariant Hirzebruch genus

$\hat{h} : U^*(pt) \to Q$

$\hat{\hat{h}} : U^*(X) \to K(X) \otimes Q$

$h^G(X) : \hat{h}(p_!(1)) \in K(pt) \otimes Q$

Borel

index $\alpha$-genus

If $G$ is connected

Atiya - Hirzebruch rigidity property

$\mathrm{Sign}(X) = \sum \mathrm{Sign}(F_s)$ $S^1$ manifold

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$h : U^* \to Q$

$h^G : U^* \to K(BG) \otimes Q$ equivariant genus

$h$ is rigid if

$(\mathrm{Im}\ h^G) \subset Q \subset K(BG) \otimes Q$ for

G - connected

Th $\left\{ \text{Elliptic genus is rigid for SU manifolds} \right.$

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Cohomology theory $\approx (U^* + \text{Hirzebruch})$

$$U \downarrow \begin{cases} Ell \\ K. \\ d. \end{cases}$$

(defined by genus with the rigidity property)