June 3, 2020

Tuesday, June 2, 2020 10:29 PM

Enumerative geometry & geometric representation theory

Start time Moscow 17:30
New York 10:30

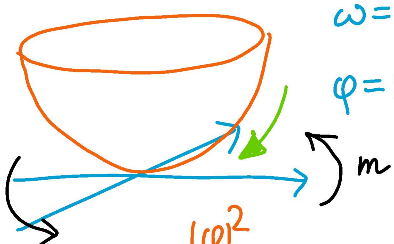
X = moduli of vacua in some very SUSY QFT in 2+1 dimensions

e.g. $T^* \mathrm{Grass}(k,n)$, $\mathrm{Hilb}(\mathbb{C}^2, k)$, ...

- wants to be hyperKähler manifold, or a holomorphic symplectic variety
- wants to be a critical locus of some function on configurations of fields / gauge transformations

perturb

one importan scenario:



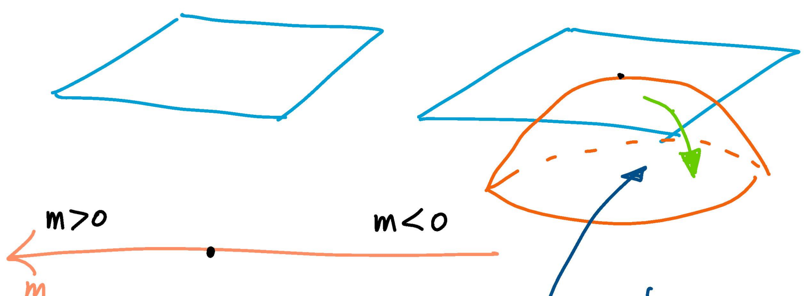
$\omega = dx \wedge dy$

$\varphi = x + iy$.

m

$\mathbb{C}$

zero function.



$m>0$ $m<0$

m

this is the action of the $GL(1) \supset U(1)$

$m \frac{|\varphi|^2}{2}$ = moment map

$\|$

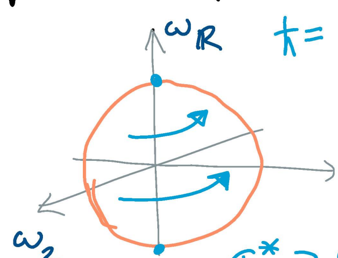
$m \in \mathrm{Lie}\, U(1)$ $\langle$ moment map, $m \rangle$

$\cap$

$\mathrm{Lie}\, U(1)^*$.

Suppose U is a compact group that acts by automorphisms of $(X, \underbrace{\omega_1}_{\omega_{\mathbb{R}}}, \underbrace{\omega_2, \omega_3}_{\omega_{\mathbb{C}}})$

$$\mu : X \longrightarrow \mathfrak{u}^* \otimes \mathbb{R}^3$$



$\omega_{\mathbb{R}}$ ω_2 ω_3

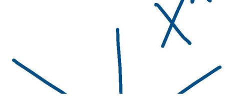
$\hbar$ = deformation parameter in $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$

$\mathbb{C}^*_\hbar \supset U(1)_\hbar$

$\langle \mu_{\mathbb{R}}, m \rangle$ = function $m \in \mathfrak{u}$

Two option: (related by 3D mirror symmetry)

① consider this U as external field



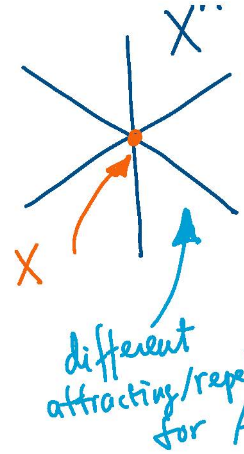
X^A $\mathfrak{u} \in \mathfrak{a}_{cp}$.

$\|$

Two option: (...

$u \in \mathfrak{u}_{cp}$ $\parallel$ $\mathrm{Lie}(A_{cp})$.

max torus in U



① consider this U as external field

$$\mathrm{Crit}_X \langle \mu_{\mathbb{R}}, m \rangle = X^m$$

Since A acts preserving $\omega_{\mathbb{C}}$ attracting and repelling directions are always paired

interface between X and X^A wants to be a holomorphic Lagrangian submanifold

which looks like the attracting manifold of X^A in X

stable envelopes = good correspondes that come as close as possible to

② add U to gauge symmetries, means

- to compute $\mathrm{Crit} \langle \mu(x), m \rangle$ in both x and m, with $m \in \mathfrak{u}^* \otimes \mathbb{R}^3$

$$\Rightarrow \quad m \cdot x = 0$$

$$\mu_{\mathbb{R}}(x) = 0 \qquad \mu_{\mathbb{C}}(x) = 0$$

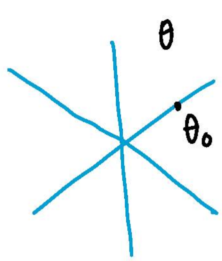
or $\mu_{\mathbb{R}}(x) = \theta \in (\mathfrak{u}^*)^{\mathfrak{u}}$

external field.

- take the quotient by U

$$X \to X /\!/\!/\!/_{\theta} U$$

hyperkähler reduction, or holomorphic symplectic reduction

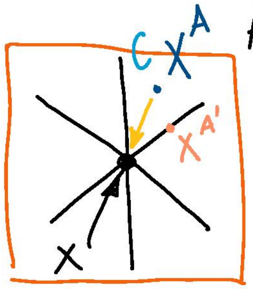


$$X_{\theta\text{-stable}} \hookrightarrow X_{\theta_0\text{-stable}}$$

we would like a good map on K-theory, $D^b\,\mathrm{Coh}$, Ell, ...

we will find out
1) not really different from stable envelopes
2) mirror dual to part ①

General problem X nonsingular alg. variety, perhaps holo sympl, ...
$A \circlearrowright X$, A = torus.



we want the nicest possible map

$$K_{eq}(X^A) \xrightarrow{\sim} K_{eq}(\mathrm{Attr}_C(X^A))$$

K_{eq}: the centraliser of A in $\mathrm{Aut}(X)$

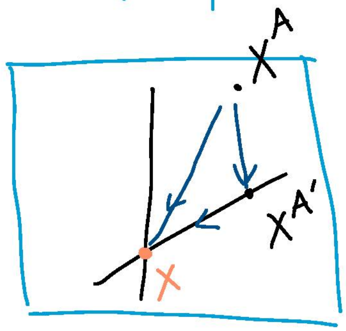
$$\mathrm{Attr}_C(X^A) = \left\{(f,x) \,\middle|\, \lim_{u\to 0} \sigma(u)\cdot x = f\right\}$$

$\sigma: \mathbb{C}^\times \to A \quad d\sigma \in C$

for $H^\bullet_{eq}$, $\mathrm{Ell}^\bullet_{eq}$, $D^b\mathrm{Coh}$, ...

In examples like $X = T^*Gr(k,n)$, $GL(\mathbb{C}^n) \supset A$

$X^a = \coprod$ varieties of the same kind.



$\Rightarrow YB$

$\Rightarrow K(X^a) = \otimes K(X''')$ gives R-matrices

Equivariant cohomology theories

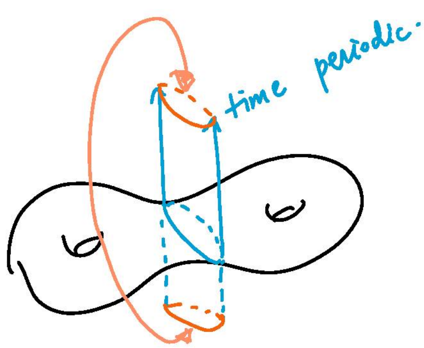
$H^\bullet_{eq}(X)$

$K_{eq}(X)$ beautiful and concrete

$\mathrm{Ell}_{eq}(X)$

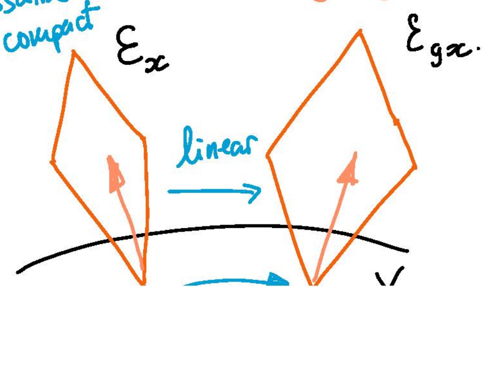
equivariant cobordism

next time a guest lecture by Igor Krichever



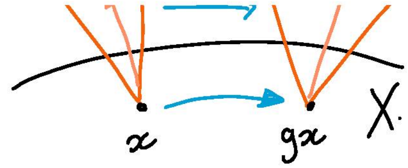
topological K-theory: compact group $G \circlearrowright X$ assume compact

$\mathcal{E}_1 \oplus \mathcal{E}_2$ and $\mathcal{E}_1 \otimes \mathcal{E}_2$



$\mathcal{E}_1 \oplus \mathcal{E}_2$ and $\mathcal{E}_1 \otimes \mathcal{E}_2$

makes the set of G-equivariant bundles a semiring



$K_G(X) =$ add $\ominus$ formal difference

pull back

$K_G(pt) =$ representation ring of G

a group homo., preserving action.

contravariant w.r.t $(G_1, X_1) \longrightarrow (G_2, X_2)$.

algebra over $K_G(pt) = \mathbb{Z}[G]^G$ for abelian groups $\mathbb{Z}[G]$

$$\operatorname{Spec} K_G(X) \longrightarrow G$$

Algebraic versions: $K_G^{perf} =$ vector bundles in alg. geometry.

G-equivariant coherent sheaves

generators

$$K_G(X) = K(\mathrm{Coh}_G(X))$$

$\mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{E}$

relations

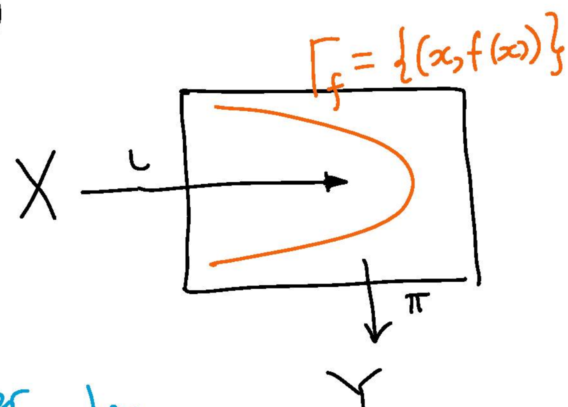
locally free $\circlearrowleft G$

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0.$$

$$[\mathcal{E}_2] = [\mathcal{E}_1] \oplus [\mathcal{E}_3]$$

if X is nonsingular than $K_G(X) = K_G^{perf}(X)$

$$f : X \longrightarrow Y.$$



$\iota_* \mathcal{E} =$ extension by zero outside Γ_f.

$$\pi_*[\mathcal{G}] \neq [\pi_* \mathcal{G}]$$

higher cohomology

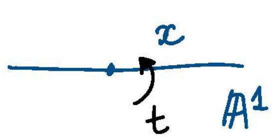
$$= \sum (-1)^i [R^i \pi_* \mathcal{G}]$$

f proper

$$= \sum (-1)^i [R^i \pi_* \mathcal{G}]$$

need to be coherent

in equivariant theory may be relaxed:



$\xrightarrow{f} pt \qquad f_* \mathcal{O}_{\mathbb{A}^1} = \mathbb{C}[x]$ has character $= 1 + t^{-1} + t^{-2} + \cdots$

suffices to assume e.g. can be contracted to a proper subset.

$\pi^* \mathcal{E}$ = pull-back.

$\mathcal{G} \otimes \mathcal{O}_\Gamma$

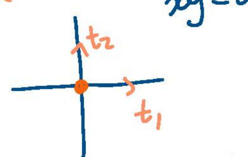
$$\iota^* [\mathcal{G}] \neq [\iota^* \mathcal{G}]$$

want to be finite

$$= \sum (-1)^i \operatorname{Tor}^i (\mathcal{G}, \mathcal{O}_\Gamma)$$

resolve either of them by locally free

has infinite resolution but this is well defined.

equivariantly may be relaxed: $X =$ 

$\mathcal{G} = \mathcal{O}_0$

can be contracted to something smooth.

excellent expositions of equivariant K-theory $\longrightarrow$ Chapter 5 of [Chriss-Ginzburg]

or [Merkuriev] in Handbook of K-theory

$$K_{GL(V)}(\mathbb{P}(V)) = ?$$

$\mathbb{P}(V) = (V \setminus 0)/GL(1)$

$\|$

$K_G(X)$

taking invariants $\downarrow$ S $\quad \uparrow$ pullback.

$K(X/G)$

also identifies all K^i

1

$$\|$$

$$K_{GL(V)\times GL(1)}(V\setminus 0)$$

center

1

$$K^1_G(X\setminus Y) \to K_G(Y) \xrightarrow{\text{push forward}} K_G(X) \xrightarrow{\text{restriction}} K_G(X\setminus Y) \to 0.$$

closed

G- invariant

2

is an isomorphism

pull back

vector space

$$\to \underbrace{K_{GL(V)\times GL(1)}}_{G}(pt) \xrightarrow[\text{push-forw.}]{} K_{GL(V)\times GL(1)}(V) \to K_{GL(V)\times GL(1)}(V\setminus 0) \to 0.$$

3.

graded modules over polynomials

$$K_{GL(V)}(\mathbb{P}(V)) = \mathrm{Coker}\left(K_G(pt) \to K_G(pt)\right).$$

inclusion of 0 in V

Koszul resolution ⇐ in 2 weeks.

next time: guest lecture by Igor Krichever.