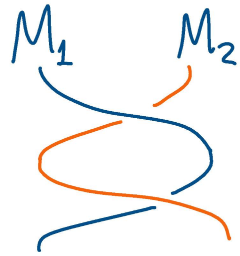
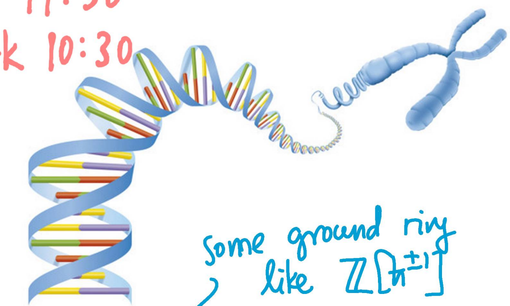
May 27, 2020

Tuesday, May 26, 2020 6:03 PM

Enumerative geometry & geometric representation theory

Start time Moscow 17:30
New York 10:30

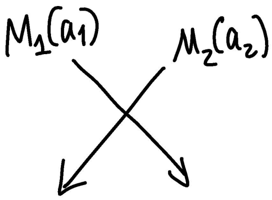




some ground ring like $\mathbb{Z}[\hbar^{\pm 1}]$

Starting point: $M_i(a)$ $\leftarrow$ vector space, a free module over $\Bbbk[a]$ or $\Bbbk[a^{\pm 1}]$

$= \text{Cohomology}_{\text{equivariant}}(X)$ $\quad a \in \mathbb{G}_m \subset \text{Aut}(X)$

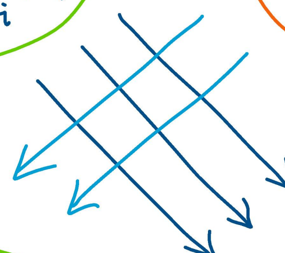


satisfies the YB equation.

Extend R-matrices to $\vec{\bigotimes} M_{k_i}(a_i)$

$\vec{\bigotimes} M_{k_i}(u_i)$ auxiliary

$\vec{\bigotimes} M_{l_i}(a_i)$ physical space



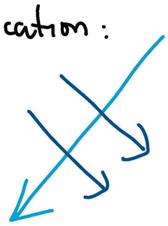
Step one:

Step two: define operators in

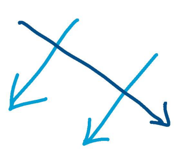
by taking matrix elements in

and values in u_i or any other linear functionals

Clearly an algebra: multiplication:



also comultiplication



needs a bit of a discussion if $\dim M_i = \infty$

e.g. for $U_\hbar(\widehat{\mathfrak{gl}}(2))$

$M = \text{Cohomology}(X)$ $\qquad X = \bigsqcup_{k \geq 0} X_k$ $\qquad X_k = T^*\mathrm{Grass}(k, \mathbb{C}^n)$

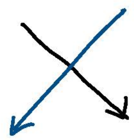
graded, with grading bounded below

R-matrices will preserve the grading

for $\mathcal{U}_\hbar(\widehat{\mathfrak{gl}(1)})$

$X_k = \mathrm{Hilb}(\mathbb{C}^2, k)$

= ideals in $\mathbb{C}[x_1, x_2]$ of codimension k



one can take graded matrix elements to define $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$

Comultiplication: $\mathcal{U}_\hbar(\hat{\mathfrak{g}}) \xrightarrow{\Delta} \mathcal{U}_\hbar(\hat{\mathfrak{g}}) \hat{\otimes} \mathcal{U}_\hbar(\hat{\mathfrak{g}})$

<u>Example</u> $\quad a \in \mathbb{G}_{add}, \mathbb{G}_{mult}$, Elliptic curve.

these are called Yangians

$u \to \infty$

$$R(u) = 1 - \frac{r}{u} + O\left(\frac{1}{u^2}\right) \qquad u \to \infty$$

finite dimensional

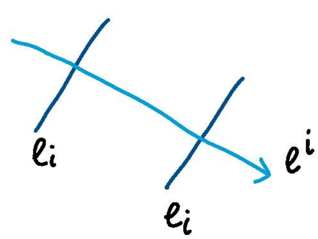
$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_\alpha \mathfrak{g}_\alpha$$

roots

$$r = \sum e_i \otimes e^i \in \widehat{S^2 \mathfrak{g}}$$

Lie algebra with a nondeg. invariant bilinear form

$\Delta e_i = ?$



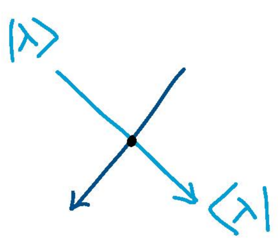
the coeff of $\left[\frac{1}{u}\right]$ is additive.

$$\Delta e_i = e_i \otimes 1 + 1 \otimes e_i$$

$[R, \Delta e_i] = 0 \qquad R$ is $\mathfrak{g}$-invariant

$|\lambda\rangle \in M$ be a lowest weight vector of weight λ $\qquad \mathfrak{g}_\alpha |\lambda\rangle = 0 \quad \alpha < 0$

$h \in \mathfrak{h} \qquad h|\lambda\rangle = \lambda(h)|\lambda\rangle$



$$R^{|\lambda\rangle, \cdot}_{\langle\lambda|, \cdot} = 1 - \frac{h_\lambda}{u} + \frac{h_\lambda^{(1)}}{u^2}$$

$\sum \lambda(h_i) h^i$

deformation of $t h_\lambda$ in our deformation $\mathcal{U}(\mathfrak{g}[t])$

[illegible] by

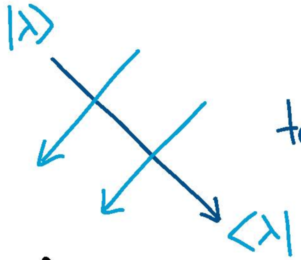
$\sum \lambda(h_i) h^i$ $\|$

act by rk(universal bundle)

these act by $c_1(-\|-)$

$\mathcal{U}(\mathfrak{g}[t])$
$\|$
$\mathrm{Maps}(\mathbb{G}_{add}, \mathfrak{g})$

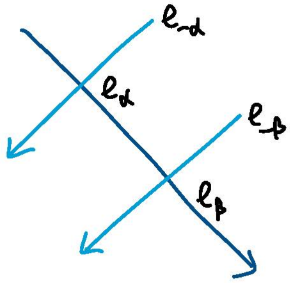
$\Delta h^{(1)}_\lambda$



take the coefficient of $\frac{1}{u^2}$

$\mathfrak{g}_{-\alpha} \otimes \mathfrak{g}_\alpha$
$\Downarrow$

$$\Delta h^{(1)}_\lambda = h^{(1)}_\lambda \otimes 1 + 1 \otimes h^{(1)}_\lambda - \sum_{\alpha > 0} (\lambda, \alpha)\, r_{-\alpha}$$



$$\langle \lambda | e_\beta e_\alpha | \lambda \rangle = \begin{cases} 0 & \text{unless } \alpha + \beta = 0,\ \alpha > 0 \\ -(\lambda, \alpha) & \text{if } \alpha + \beta = 0,\ \alpha > 0. \end{cases}$$

$[e_{-\alpha}, e_\alpha] = -h_\alpha$

Two remarks: ① Usually, the condition that $[R, \Delta \mathfrak{g}] = 0$ and

$$R \, \Delta h^{(1)}_\lambda = \Delta_{opp} h^{(1)}_\lambda \, R$$

determine R uniquely up to a multiple.

② Example $\mathfrak{g} = \widehat{\mathfrak{gl}(1)} \ni \alpha_n$

central

$[\alpha_n, \alpha_m] = n \, \delta_{n+m} \, c$

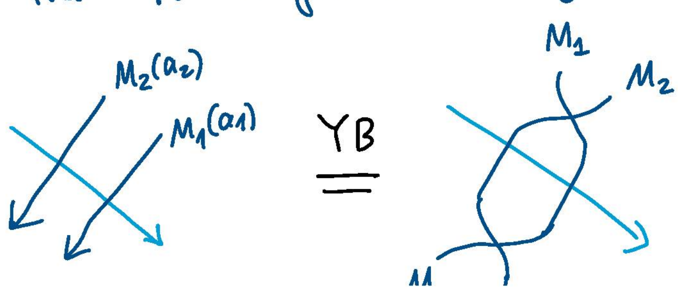
$r = \ldots + \sum_n \alpha_n \otimes \alpha_{-n}$

bilinear expressions in α_n ⇝ Sugawara-like formulas for Virasoro

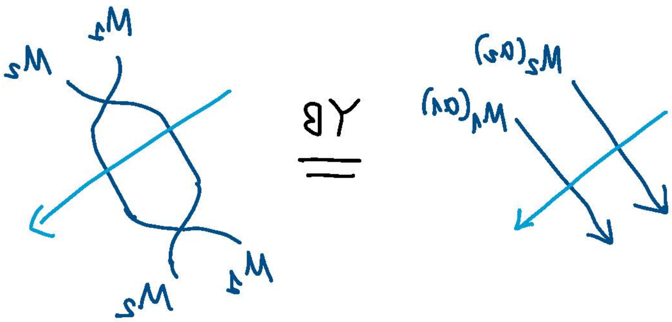
$\alpha^{(1)}_n$ = Virasoro

means R is the Liouville reflection operator

It is obvious that R do give the braiding of the modules we constructed



We can use in auxiliary space instead of the physical space!



YB is both the commutation relation and cocommutation relation!

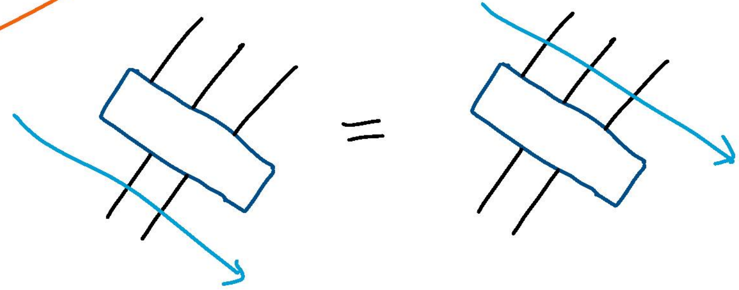
We constructed a tensor category

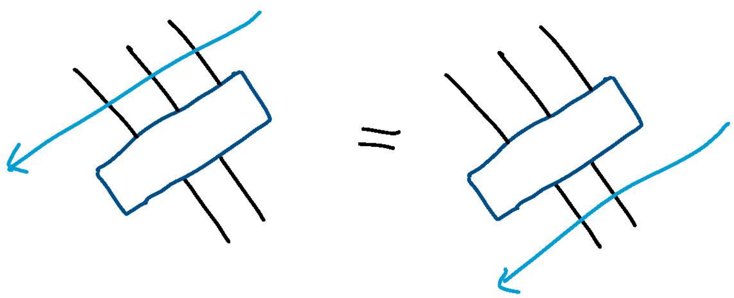
modulo poles see below

Objects $= \otimes M_{k_i}(a_i)$

Morphisms = maps that commute with R-matrices

also a relation in $U_\hbar(\hat{g})$





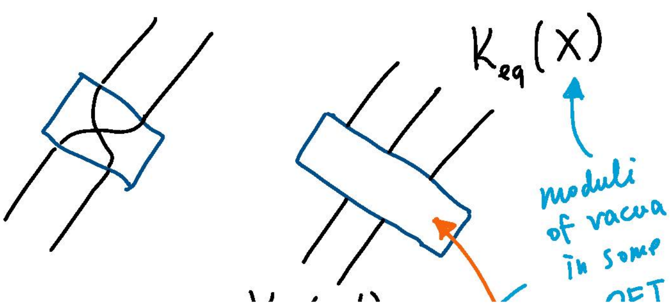
reductive

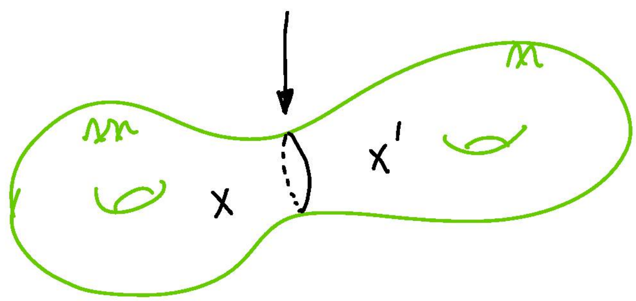
$G \subset GL(V)$ H.Weyl

$\Updownarrow$

$\mathrm{Hom}_G(V^{\otimes k} \longrightarrow V^{\otimes l})$

How to actually construct R-matrices.?





of vacua in some QFT

$K_{eq}(X')$

variation of parameters of QFT

Interface new physics

more physics here

a very typical scenario in equivariant cohomology !

$M_2(a_2)\otimes M_1(a_1)$ $\qquad$ $M_1(a_1)\otimes M_2(a_2)$

$\longrightarrow a_1/a_2 \in \mathbb{C}$

$M_1(a_1) \circledast M_2(a_2)$ both embed and the Coker of either embedding is torsion

annihilated by $(1-a_1/a_2\hbar^{\cdot\cdot})^N$

a kindergarten illustration:

$$X = pt \circlearrowleft \mathbb{C}^* \ni a$$

$$K_{eq}(X) = \mathbb{Z}[a^{\pm 1}]$$

$Y = \mathbb{C} \circlearrowleft \mathbb{C}^* \ni a$ defining.

$Y \downarrow X$ pullback gives

$$K_{eq}(Y) = \mathbb{Z}[a^{\pm 1}]$$

pushforward $\quad K_{eq}(X) \longrightarrow K_{eq}(Y)$

$$0 \to x\,\mathbb{C}[x] \longrightarrow \mathbb{C}[x] \to \mathcal{O}_0 \to 0$$

has weight $= a^{-1}$ $\qquad$ coordinate $\qquad$ $K_{eq}(X)$

$$\mathcal{O}_0 = (1-a^{-1})\,\mathcal{O}_{\mathbb{C}}$$

$$\boxed{\mathrm{Coker}\left(K_{eq}(X) \to K_{eq}(Y)\right) = \mathbb{Z}[a^{\pm 1}]/(1-a^{-1}).}$$

Key example. $\mathcal{U}_\hbar(\widehat{\mathfrak{gl}(2)})$ on $K_{eq}(X_n)$, $\quad X_n = \bigsqcup_k T^*\mathrm{Gras}(k,n)$

scaled by $\hbar^{-1}$ $\qquad$ $\circlearrowleft \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \in GL(n)$

$$X_1 = pt \sqcup pt$$

$$K_{eq}(X_1) = \mathbb{Z}[\hbar^{\pm 1}, a_1^{\pm 1}]^{\oplus 2}$$

$\parallel$

$K_{eq}(X_1) = \mathbb{Z}[\hbar^{\pm}, a_1^{\pm}]$

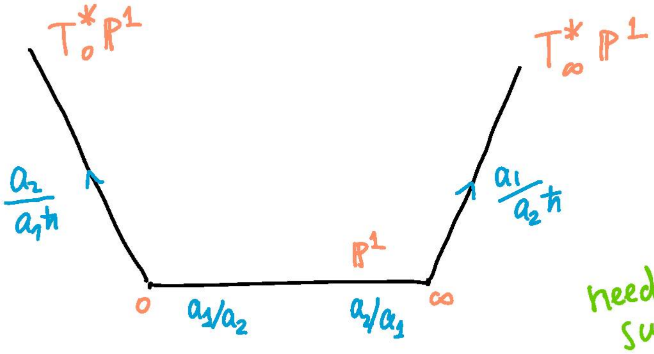
$\cong$ "$\mathbb{C}^2(a_1)$"

$\otimes_{opp.} \longrightarrow K(X_2) \longleftarrow \mathbb{C}^2(a_1)\otimes\mathbb{C}^2(a_2)$

$X_2 = pt \sqcup T^*\mathbb{P}^1 \sqcup pt$

by the weight of f.

$T^*\mathbb{P}^1$

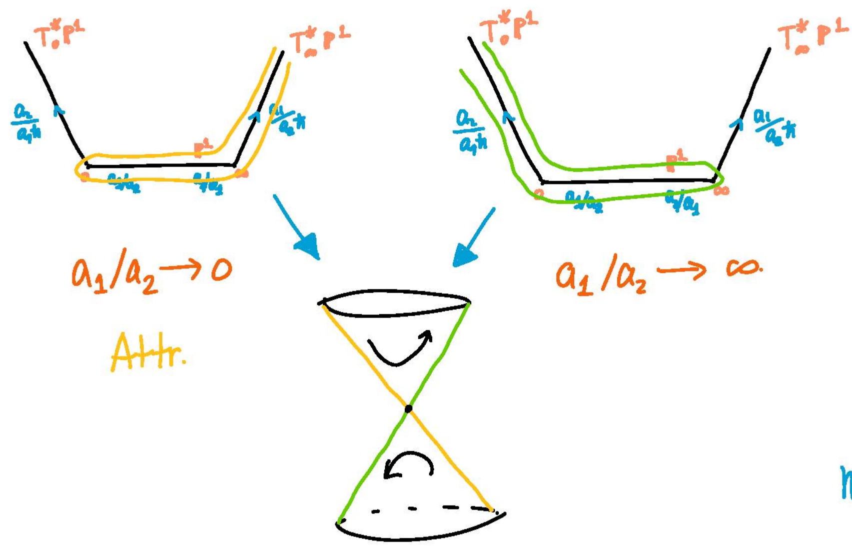


$K_{eq}(2pt) \longrightarrow K_{eq}(T^*\mathbb{P}^1) \longleftarrow$

$K_{eq}(\mathrm{Attr}_{a_1/a_2\to 0})$

need to supply!

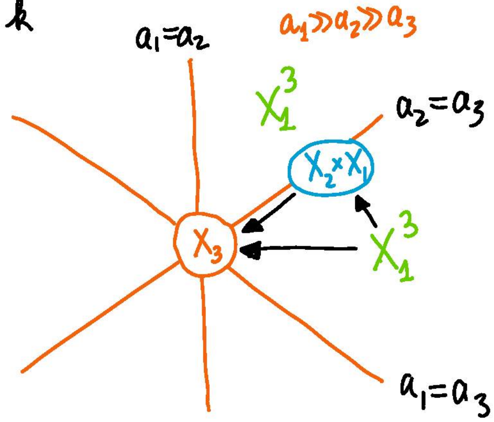
different in different cohomology theory



not unique.

needs to satisfy the following analog of the YB equation

$X_3 = \bigsqcup_k T^*\mathrm{Grass}(k,3)$



for every a, consider X^a

need to commute!!

Next time: intro to $H^\bullet_{eq}$, K_{eq}