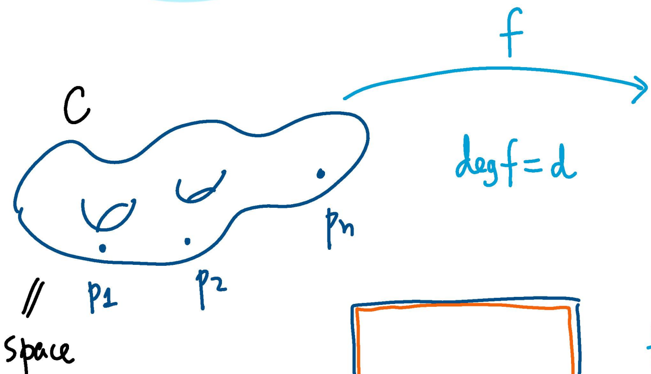
May 20, 2020

Tuesday, May 19, 2020 8:25 PM

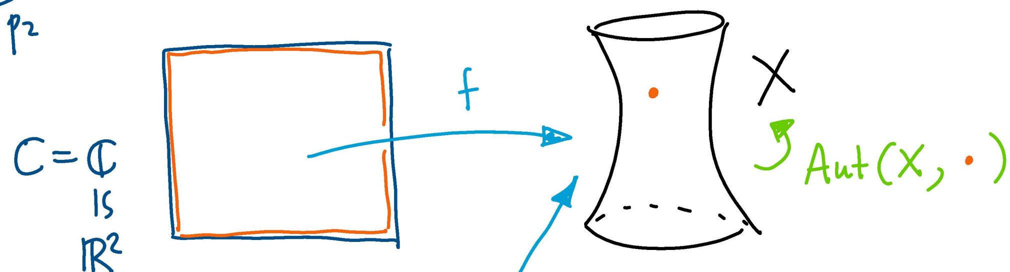
Enumerative geometry & geometric representation theory

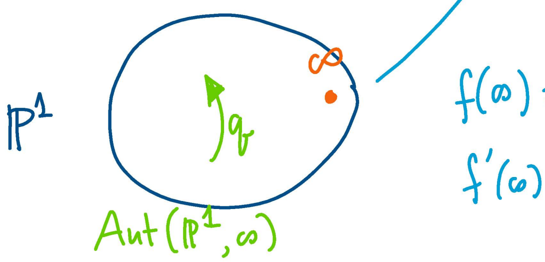
Start time Moscow 17:30

New York 10:30



X = (a component of) the moduli of vacua





$f(\infty)$ = given pt in X

$f'(\infty)$ finite

open set in the space of maps

pushforward = take sections of a holo vector bundle on a open set

↑ ∞-dimensional vector space

character of $\mathrm{Aut}(\mathbb{P}^1, \infty) \times \mathrm{Aut}(X, \cdot)$

is well-defined is a rational function

for given degree $d = \deg f$

character of the torus $H^2(X, \mathbb{Z}) \otimes \mathbb{C}^\times$

$$\sum_d z^d \cdot \text{rational function}(q, \dots)$$

= a fancy generalization of the q-hypergeometric function

$$\Phi\left[\begin{matrix} a & b \\ & c \end{matrix} \middle| z, q\right] = \sum_d z^d \frac{(a)_d\,(b)_d}{(q)_d\,(c)_d}$$

$$(x)_d = (1-x)(1-qx)\dots(1-q^{d-1}x)$$

$$= \frac{(x)_\infty}{\cdots}$$

$$\phi\left[\begin{matrix} c \end{matrix}\middle| t, q\right] = \sum_d \cdots \frac{\cdots}{(q)_d\,(c)_d} \qquad = \frac{(x)_\infty}{(q^d x)_\infty}$$

Solutions of regular q-difference in $\boxed{z}$

and also a q-difference equation in $\mathrm{Aut}(X)$-variables which is regular in $\boxed{\mathrm{Aut}(X,\omega)}$

Duality $X \longleftrightarrow X^\vee$ exchanges

partition

vector $= (x_1, \dots, x_n)$

Example Macdonald polynomial for $GL(n)$ $P_\lambda(x; q, t)$ a terminating case of a certain hypergeometric function for

$$(a_1, \dots, a_n) = (q^{\lambda_1} t^{n-1}, \dots, q^{\lambda_n})$$

label/argument symmetry of Macdonald poly equations in a $=$ equations in x

this is an example of the general theory for $X = T^* GL(n)/B = X^\vee$

these q-difference eq. for $GL(n)$ Macdonald polynomials are a special case of so called dynamical equation of Etingof & Varchenko for $\mathcal{U}_\hbar(\mathfrak{g}_{KM})$

we will need to generalize that

¿ What is a quantum group?

Say, let $G \ni g$ be a finite group $\longrightarrow$ $\mathbb{K}G \ni \sum c_g g$

a field (maybe ring)

$$\mathbb{C}G = \bigoplus_{\substack{\text{irreps } M \\ \text{of } G}} \mathrm{End}_{\mathbb{C}}(M)$$

the category of G-modules

has (1) tensor product $(M_1, M_2) \longrightarrow M_1 \otimes M_2$ with unit $\mathbb{1}$

$g \otimes g$

coassociative

$$\Delta(g) = g \otimes g$$

has (1) tensor product (M_1, M_2)

with unit $\mathbb{1}$ $\qquad g \otimes g$

(2) duals $M^\vee = \mathrm{Hom}(M, \mathbb{k})$

$(g^{-1})^T$

$\Delta(g) = g \otimes g$

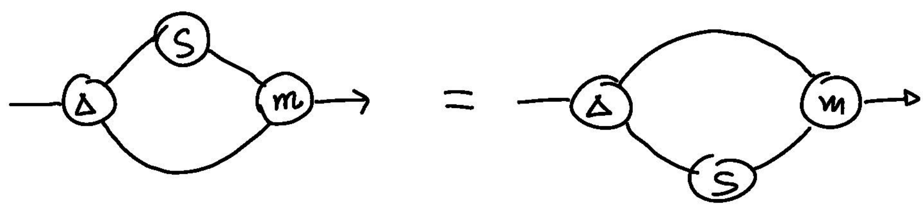
homomorphis of algebras $\mathbb{k}G \xrightarrow{\Delta} \mathbb{k}G \otimes \mathbb{k}G$

trivial representation $\mathbb{1}$: $\mathbb{k}G \to \mathbb{k}$

counit

antipode $S : \mathbb{k}G \ni g \longrightarrow g^{-1} \in \mathbb{k}G$ antiautomorphism

satisfy natural relations $\quad gg^{-1} = g^{-1}g = 1$ $\quad$ means



= unit ∘ counit

these are axioms for a Hopf algebra ↻ duality $\mathbb{k}G = \mathbb{k}[G]$

function on G

m = multiplication of functions

$\Delta f(g_1, g_2) = f(g_1 g_2)$.

if G is a Lie group

$\mathbb{k}G$	$\mathbb{k}[G]$
$\mathbb{k}G$	$\mathbb{k}[G]$ functions in whatever category
(signed) measures on G	
distributions on G	$H^*(G, A)$ ⇐ original case investigated by Hopf
∪	
$\mathcal{U}(\mathfrak{g})$ $\quad \Delta\xi = \xi \otimes 1 + 1 \otimes \xi$	commutative
$H_*(G, A)$	
cocommutative	

A quantum group is a deformation of any of these in the world of Hopf algebras

$$\boxed{\Delta \neq (12)\Delta}$$

$(12)\Delta = \Delta_{opp}$

in $\mathcal{U}(\mathfrak{g})$ we have $xy - yx = [x, y]$ in $\mathfrak{g}$

$xy - yx$ // $m - m \circ (12)$

Δ and Δ will be related by a cocommutation relation

Δ and Δ_{opp} will be related by a cocommutation relation

Concretely, we want $\mathcal{U}(\hat{\mathfrak{g}}) \rightsquigarrow \mathcal{U}_\hbar(\hat{\mathfrak{g}})$ where

$$\hat{\mathfrak{g}} = \mathrm{Maps}\,(\mathbb{G} \longrightarrow \mathfrak{g})$$

commutation taken pointwise

$\mathfrak{g}[t]$ for $\mathbb{G} = \mathbb{G}_{add}$

$\mathfrak{g}[t^{\pm 1}]$ $\mathbb{G} = \mathbb{G}_{mult}$.

$\mathbb{G} \curvearrowright \hat{\mathfrak{g}}$ loop rotation

the additive group, the mult. group, or E ...

The category of modules that we want to study deforms

$$\hat{\mathfrak{g}} \xrightarrow[\text{at } a]{\text{evaluate}} \mathfrak{g} \curvearrowright \text{highest weight module } M$$

combined module denote $M(a)$

$\bigotimes_i M_i(a_i)$ different points a_i dont talk to each other

in particular irreducible if M_i is irreducible and $a_i \neq a_j$

upon deformation $M_1(a_1) \otimes M_2(a_2) \longrightarrow M_1(a_1) \otimes_{opp} M_2(a_2)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \parallel$

$R_{M_1, M_2}(a_1, a_2)$ is an isomorphis for a_1 and a_2 <u>generic</u>

a rational function of a_1 and a_2 with $\det \neq 0$.

the loop rotation automorphism survives only depends on $a_1/a_2 =: u$

<u>Our strategy</u> first construct R! The rest, including $\mathfrak{g}$, would follow

geometrically

Our R will satisfy the following relations: $\qquad$ $R^\vee$

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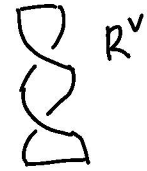


$$M_1 \otimes M_2 \xrightarrow[\;R^\vee\;]{(12)\,R} M_2 \otimes M_1 \xrightarrow{R^\vee} M_1 \otimes M_2$$

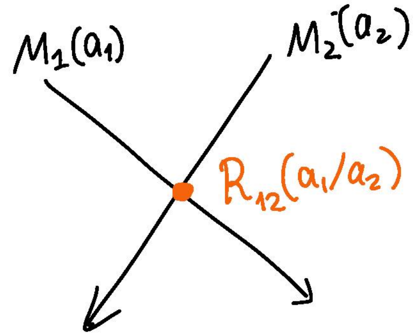
appears in works on R matrices without spectral parameter $R(0)$ or $R(\infty)$

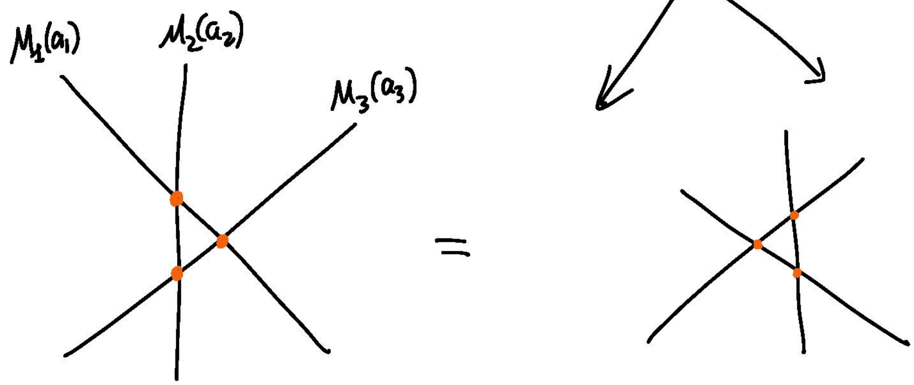
$$(R^\vee)^2 = 1 \qquad R_{21}(u^{-1})\,R_{12}(u) = 1.$$

$$M_1 \otimes M_2 \otimes M_3 \rightrightarrows M_3 \otimes M_2 \otimes M_1.$$

two different ways to put in the opposite order are simply equal

Yang-Baxter equation





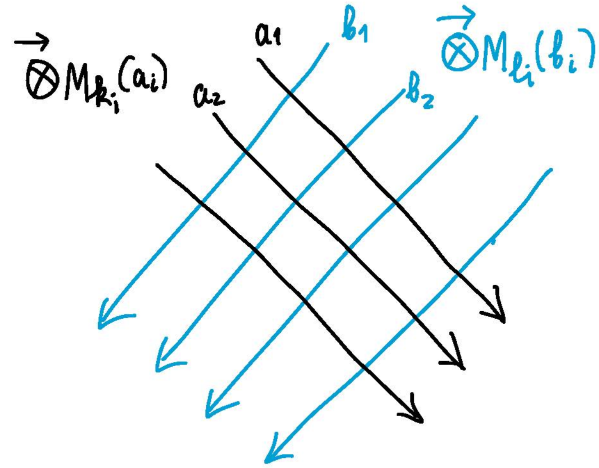
This is the data that will be constructed geometrically

Remarkable fact: from R-matrices $\rightsquigarrow$ makes $\otimes M_{k_i}(a_i)$ modules over a certain quantum group

Reshetikhin "Quasitriangular Hopf algebras & invariants of links" Algebra i Analiz vol 1 (1989).

also books by Etingof ...

Steps ① extend $R_{M_2, M_2}(a_1/a_2)$ to

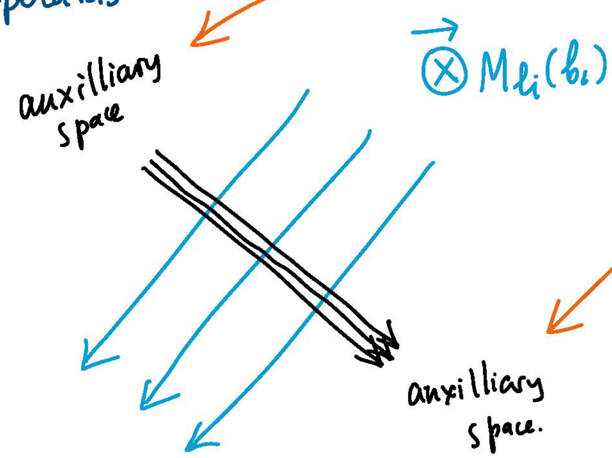


product of 3×4 R-matrices satisfies YB

(2) extend to duals ....

(3) define operators

take matrix coeff. here.



big generating function of operators