May 13, 2020

Enumerative geometry & geometric representation theory

Start time Moscow 17:30
New York 10:30

Old problem in repr. theory $G \hookrightarrow$ manifold M (iso), $M \xrightarrow{\varphi} \mathbb{R}$ (G-invariant)

Eigenspaces of $(-\Delta + \varphi)$ as G-modules

in particular $\operatorname{tr} \operatorname{evolution}(T) \cdot g = ?$

Very good cases $M = G/H$ or $M = \mathbb{R}^n$, $\varphi = \varphi(r)$

$\circlearrowleft$

$G = O(n)$

great simplification in the presence of supersymmetry

$$\mathcal{H} = \mathcal{H}_{even} \underset{D}{\oplus} \mathcal{H}_{odd} = \text{Sections}(M, V_{\bar{0}} \oplus V_{\bar{1}})$$

Hilbert space

even odd

$-\Delta + \varphi = (D + D^*)^2 \qquad D^2 = 0$

e.g. $\Omega^{even/odd} M$

e.g. $D = d_{\text{de Rham}}$

$$\mathcal{H} = \bigoplus \mathcal{H}^{\lambda}$$

eigenvalue

for $\lambda \neq 0$

$$\mathcal{H}^{\lambda}_{even} \xrightarrow{D} \mathcal{H}^{\lambda}_{odd} \xrightarrow{D} \mathcal{H}^{\lambda}_{even}$$

is an exact sequence of G-modules

$\mathcal{H}^{\lambda}_{even} \ominus \mathcal{H}^{\lambda}_{odd} = 0 \in$ representation ring of G $\oplus, \otimes$ = ring generated by characters of G

e.g. $G = U(n)$ $\quad \mathbb{Z}[x_1, \dots, x_n]^{S(n)}$

for $\lambda = \underline{\underline{0}}$ ground states of our Hamiltonian

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$$[\mathcal{H}^0_{even}] - [\mathcal{H}^0_{odd}] \in \text{representation ring} =: Rep(G)$$

$$\overset{\text{def}}{=} \text{Index} \overset{\text{"="}}{=} \mathcal{H}_{even} - \mathcal{H}_{odd}$$

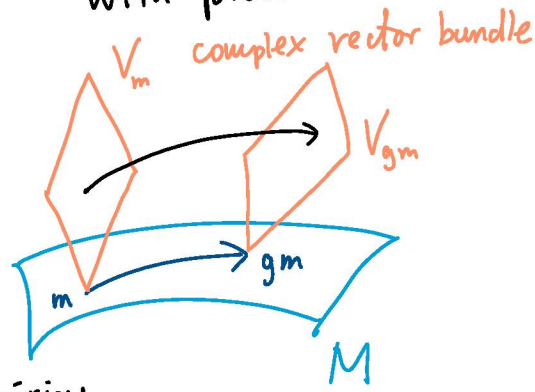
$$\mathrm{str}_{\mathcal{H}} \text{evolution} \cdot g = \mathrm{tr}_{\text{index}}\, g$$

Topological invariant of an elliptic differential operator ← given by AS formula in term of G-equivariant K-theory of M.

Top. space $M \circlearrowleft G$ $\xrightarrow{\text{functor}}$ $K^i_G(M)$ abelian groups with product

is an example of a cohomology theory if. M is compact

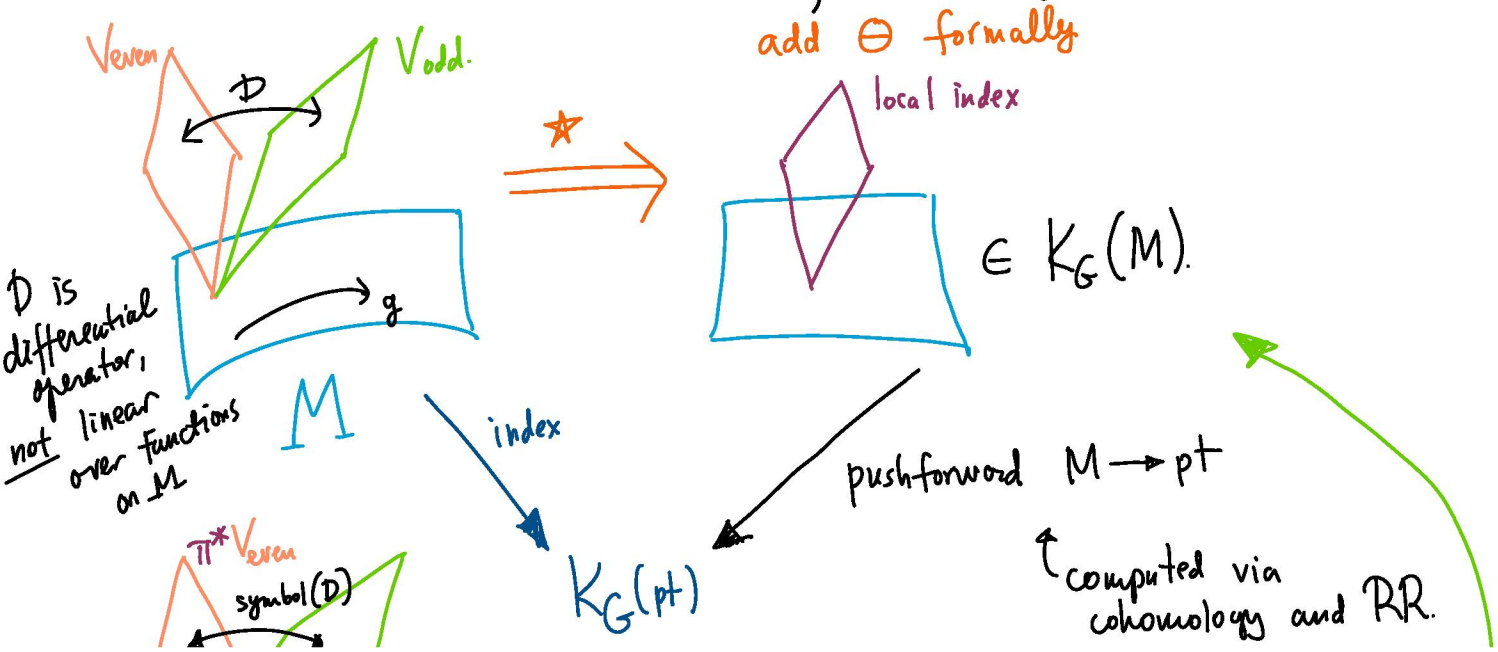
$K^0_G(M) \ni$

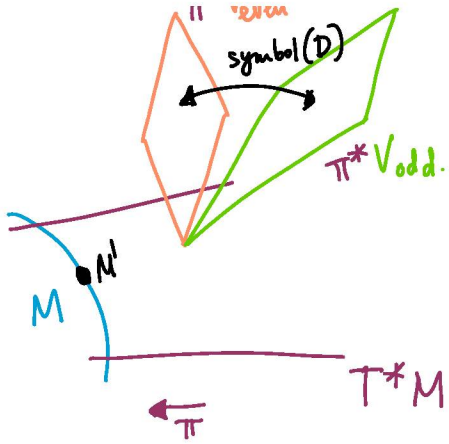


$$K^0_G(pt) = Rep(G)$$

$\oplus, \otimes$ semiring

add $\ominus$ formally





$K_G(pt)$ (computed via cohomology and RR.)

elliptic means that this is an iso away from zero section $M \subset T^*M$

means comes from local index $\in K_G(M)$

Question: maybe it comes from something even smaller? $M' \subset M$

eg.. tr g. evolution(T) comes from $M^g \subset M$

Really important if M is ∞-dimensional QFT in $(d+1)$ dim $= (2+1)$

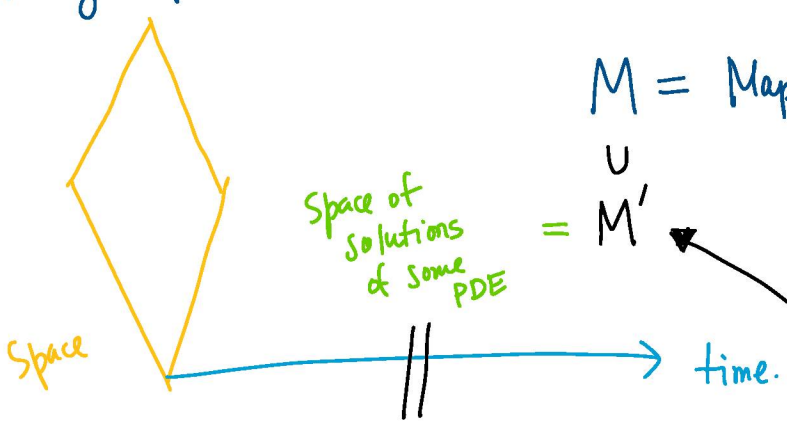
$$M = \mathrm{Maps}(\text{space} \to X)$$

for now, the configuration space at a point

$$\cup$$

$$= M'$$

Space of solutions of some PDE



$\sqcup$ finite dimensional spaces

those come from extended supersymmetry brothers of D

$$M = \mathrm{Holo}((C, p_1, \dots, p_n) \xrightarrow{f} X)$$

compact Riemann surface

marked pts

alg. variety

On such M one constructs by means of alg. geometry some element $K_G(M)$ that stands in for the local index of Dirac operator D

of Dirac operator D

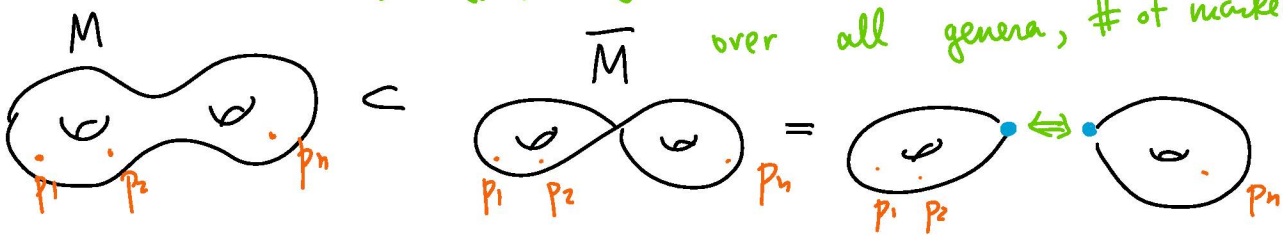
forget f except $f(p_1),\dots,f(p_n)$

$$K_G\big(\mathrm{Moduli}(C,p_1,\dots,p_n)\times X^n\big) \qquad G=\mathrm{Aut}(C,p_1,\dots,p_n)\times \mathrm{Aut}(X)$$

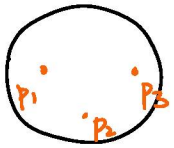
very nice language to talk about K classes here.

CohFT

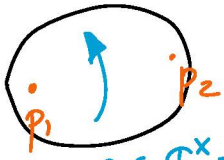
because these form a coherent collection over all genera, # of marked pts, ...



reduces to a few basic tensors like



$\in K_{\mathrm{Aut}(X)}(X^3)$



$\in K_{\mathbb{C}^*_q\times \mathrm{Aut}(X)}(X^2)$

$q\in\mathbb{C}^\times=\mathrm{Aut}(\mathbb{P}^1,0,\infty)$.

<u>1st goal</u>: explain how to compute these in terms of some geometric representation theory for which $K_{eq}(X)$ are spaces

of certain quantum loop groups

<u>Silly example:</u>

$$X=\{0,1\}=2pt$$

$$K(X)=\mathbb{Z}^2 \circlearrowleft \mathrm{Mat}(2\times 2,\mathbb{Z})=K(X\times X)$$

is the local configuration space for e.g. $\mathfrak{sl}_2$-vertex models

the quantum group of lattice models.

$K(X)\otimes_{\mathbb{Z}}\mathbb{C}$ is module for $U_q(\mathfrak{gl}_2)\leftarrow U_q(\widehat{\mathfrak{gl}}_2)$

Important point: this $X=X_{\mathbb{R}}$ space of states defined microscopically

[... index computation one can assume that C is very large

Important point: this $\Lambda = \Lambda_{\mathbb{R}}$

for index computation one can assume that C is very large

moduli of vacua

$X_{UV} =$  $= [0,1]^2 =$ average density of horizontal and vertical paths.

$\|$

vacuum states in $\mathbb{R}^2$ = states in $\mathbb{R}^2 \times \mathbb{R}$ = Gibbs states

infinite space infinite time.

In this course we will focus on $X = X_{\mathbb{R}}$ with even more supersymmetry

namely it will be algebraic symplectic

G will <u>scale</u> the symplectic form ω_X

the corresponding character $\hbar : G \to \mathbb{C}^\times \in K_G(pt)$ will be the deformation parameter in $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$

<u>Amazing phenomenon</u>: 3d mirror symmetry

(if X is a Lie algebra of XXI century then this is the Langlands duality)

$$X \longleftrightarrow X^\vee$$

index $\in K_{eq}\left(\text{Moduli}(\text{  }) \times X^n\right)[[z]] \overset{!}{=}$ same for $X^\vee$ with.

$z \longleftrightarrow$ maximal torus $A^\vee$ in $\mathrm{Aut}(X^\vee, \omega^\vee)$

variable that remembers the degree of the map f:  $\to X$

$A \longleftrightarrow Z^\vee$

$d = \deg(f) = f_*([\text{  }]) \in H_2(X, \mathbb{Z})$.

z^d = character of torus Z dual to

poles in A and Z are <u>roots</u>

equivariant Kähler

usual Langlands $X = T^*G/B$

$A \subset G$ max torus

Optimistic goal: prove this whenever defined

long process

<u>Next time</u>: introduction to quantum groups.