May 13, 2020

Enumerative geometry vs geometric representation theory

Start time
Moscow 17:30
New York 10:30

Old problem in repr. theory
$G \hookrightarrow \text{manifold } M \xrightarrow{\text{iso}} \text{IR}^n \xrightarrow{\text{G-invariant}}$

Eigenvalues of $(-\Delta + \varphi)$ as $G$-modules
in particular $\text{tr} \text{evolution} (T) \cdot g = ?$

Very good cases $M = G/H$ or $M = \text{IR}^n \xrightarrow{\varphi = \varphi(r)} G = \text{O}(n)$

Great simplification in the presence of supersymmetry

$\mathcal{H} = \mathcal{H}_{\text{even}} \oplus \mathcal{H}_{\text{odd}} = \text{Sections}(M, V_0 \oplus V_1)$

Hilbert space $-\Delta + \varphi = (D + D^*)^2 \quad D^2 = 0$

e.g. $\Omega$ even/odd $M$
e.g. $D = d_{\text{de Rham}}$

$\mathcal{H} = \bigoplus \mathcal{H}^\lambda$ for $\lambda \neq 0$$\mathcal{H}^\lambda \xrightarrow{D} \mathcal{H}^\lambda \xrightarrow{D} \mathcal{H}^\lambda$

$\mathcal{H}_{\text{even}} \otimes \mathcal{H}_{\text{odd}} = 0 \in \text{representation ring of } G$

$\otimes, \oplus = \text{ring generated by characters of } G$
e.g. $G = \text{U}(n)$ $\mathbb{Z}[x_1, \ldots, x_n] \text{S}(n)$

for $\lambda = 0$ ground states of our Hamiltonian
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$$[H_{\text{even}}] - [H_{\text{odd}}] \in \text{representation ring} =: \text{Rep}(G)$$

$\text{Index} = H_{\text{even}} - H_{\text{odd}}$

Topological invariant of an elliptic differential operator

Top. space $M$ \xrightarrow{\text{functor}} $K^i_G(M)$

is an example of a cohomology theory if $M$ is compact

$K^0_G(M) \in K^0_G(pt) = \text{Rep}(G)$

$\oplus, \otimes$ semiring

$\ast$ local index

$\text{Push-forward } M \to pt$

$\text{computed via cohomology and RR}$

$\nabla$ is differential operator, not linear over functions on $M$

$\text{Symbol}(\nabla)$
\[ K_G(\text{pt}) \]

elliptic means that this is an iso away from zero section \( M \subset T^* M \)

means comes from local index \( \in K_G(M) \)

\( \text{comped via cohomology and RR.} \)

Question: maybe it comes from something even smaller? \( M' \subset M \)

eg. tr g. evolution \( (T) \) comes from \( M^g \subset M \)

Really important if \( M \) is \( \infty \)-dimensional QFT in \((d+1)\) dim

\[ M = \text{Maps (space } \rightarrow X) \]

\[ M' = \text{Space of solutions of some PDE} \]

\[ \text{U finite dimensional spaces} \]

\[ \text{Those come from extended supersymmetry brothers of } \mathcal{D} \]

\[ M = \text{Holo } (\mathcal{C}, p_1, \ldots, p_n) \rightarrow X \]

on such \( M \) one constructs by means of alg. geometry some element \( K_G(M) \) that stands in for the local index of Dirac operator \( \mathcal{D} \).
forget $f$ except $f(p_1), \ldots, f(p_n)$

\[
K_G(\text{Moduli}(C, p_1, \ldots, p_n) \times X^n) \quad G = \text{Aut}(C, p_1, \ldots, p_n) \times \text{Aut}(X)
\]

Coh FT

\[
\begin{array}{c}
\text{very nice language to talk about K classes here.} \\
\text{because these form a coherent collection} \\
\text{over all genera, \# of marked pts},
\end{array}
\]

\[
M \subset \overline{M} = \bigcup_{\text{all genera}} = (M, p_1, \ldots, p_n) \iff (M, p_1, \ldots, p_n).
\]

reduces to a few basic tensors like

\[
\begin{array}{c}
\bigcirc_{p_1, \ldots, p_n} \in K_{\text{Aut}(X)}(X^3) \\
\bigcirc_{p_1, \ldots, p_n} \in K_{\mathbb{C}^* \times \text{Aut}(X)}(X^2) \\
q \in \mathbb{C}^* = \text{Aut}(\mathbb{C}^*)
\end{array}
\]

1st goal: explain how to compute these in terms of some geometric representation theory for which $K_{eq}(X)$ are spaces

Silly example:

\[
X = \{0, 1\} = 2 \text{pt}
\]

\[
K(X) = \mathbb{Z}^2 \subset \text{Mat}(2 \times 2, \mathbb{Z}) = K(X \times X)
\]

is the local configuration space for e.g. $\mathfrak{g}_2$ - vertex models

\[
K(X) \otimes_{\mathbb{Z}} \mathbb{C} \text{ is module for } U_0(\mathfrak{g}_2) \leftarrow U_1(\mathfrak{g}_2)
\]

Important point: this $X = X_{\text{IR}}$ space of states defined microscopically

In a weak computation one can assume that $C$ is very large...
polar in A and A* are roots

equivalent

roots

usual Langlands \( X = T \cdot G/B \)

A \rightarrow G \rightarrow \text{max} \text{ torus}

\( Z \leftarrow Z \rightarrow \text{Aut}(X, \omega) \text{ maximal torus} \ A^V \)

\( \text{in } \text{Aut}(X, \omega) \text{ with } \Omega \in \text{Aut}(X, \omega) \text{ of type } \Omega \text{ in } \text{Aut}(X, \omega) \text{ of type } \Omega \text{ in } \text{Aut}(X, \omega) \text{ of type } \Omega \text{ in } \text{Aut}(X, \omega) \text{ of type } \Omega \text{ in } \text{Aut}(X, \omega) \text{ of type } \Omega \text{ in } \text{Aut}(X, \omega) \text{ of type } \Omega \)
Optimistic goal: prove this whenever defined long process

Next time: introduction to quantum groups.