Notes on dimers, Kasteleyn, Grassman integral & scaling limit.

- det(Kasteleyn) = matrix element of vertex operators
- matrix element can be computed
- Correlation functions & double integrals
- Scaling limit

② Vertex operators

(i) Fermionic Fock space, $\langle v_m \rangle$ in $\mathbb{C}^{\mathbb{Z}+\frac{1}{2}}$

$$F = \left\{ v_{m_1} \wedge v_{m_2} \wedge \cdots \;\middle|\; \begin{array}{l} m_i \in \mathbb{Z}+\frac{1}{2}, \\ m_{i+1} = m_i - 1, \quad i \gg 1 \end{array} \right\}$$

(ii) Clifford algebra:

$$Cl_{\mathbb{Z}} = \langle \psi_m, \psi_m^*, \; m \in \mathbb{Z}+\tfrac{1}{2} \rangle$$

$$\psi_m \psi_{m'} + \psi_{m'} \psi_m = \psi_m^* \psi_{m'}^* + \psi_{m'}^* \psi_m = 0$$

$$\psi_m \psi_{m'}^* + \psi_{m'}^* \psi_m = \delta_{m,m'}$$

(iii) It acts on F:

$$\psi_m v_{m_1} \wedge v_{m_2} \wedge \cdots = v_m \wedge v_{m_1} \wedge v_{m_2} \wedge \cdots$$

$$\psi_m^* v_{m_1} \wedge v_{m_2} \wedge \cdots = \sum_{i=1}^{\infty} (-1)^i \delta_{m_i m} \cdot$$

$$\cdot \, v_{m_1} \wedge \cdots \wedge \widehat{v_{m_i}} \wedge \cdots$$

(iv) Heisenberg algebra:

$$\langle \alpha_n \mid n \in \mathbb{Z} \setminus \{0\} \rangle$$

$$[\alpha_n, \alpha_{n'}] = -n\, \delta_{n,-n'}$$

(v) It acts on F

(part of Bose-Fermi correspondence in 1D)

$$\alpha_n \mapsto \sum_{m \in \mathbb{Z}+\frac{1}{2}} \psi_{m+n} \psi^*_m$$

As operators in F:

$$[\alpha_n, \psi_k] = \psi_{k+n}$$

$$[\alpha_n, \psi^*_k] = -\psi^*_{k-n}$$

(vi) Vertex operators (in F):

$$\Gamma_{\pm}(x) = \exp\left(\sum_{n=1}^{\infty} \frac{x^n}{n} \alpha_{\pm n} \right)$$

$$(\Gamma_-(x)v, w) = (v, \Gamma_+(x)w) = (\Gamma_+(x)w, v)$$

(iii) Commutation relations

$$\Gamma_+(x)\Gamma_-(y) = (1-xy)\Gamma_-(y)\Gamma_+(x)$$

$$\Gamma_+(x)\psi(z) = (1-z^{-1}x)^{-1}\psi(z)\Gamma_+(x)$$

$$\Gamma_-(x)\psi(z) = (1-xz)^{-1}\psi(z)\Gamma_-(x)$$

$$\Gamma_+(x)\psi^*(z) = (1-z^{-1}x)\psi^*(z)\Gamma_+(x)$$

$$\Gamma_-(x)\psi^*(z) = (1-zx)\psi^*(z)\Gamma_-(x)$$

(iv) eigenvectors:

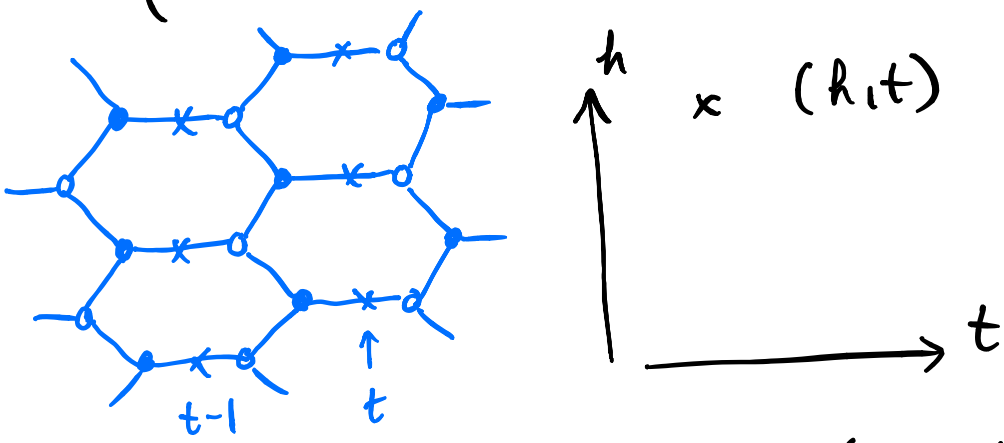
$$\Gamma_-(x)\prod_i \psi^*(w_i)\prod_j \psi(z_j)v_0^{(n)} =$$

$$= \prod_i (1-xz_i)^{-1}\prod_j (1-xw_j)\prod_i \psi^*(w_i)\prod_j \psi(z_j)v_0^{(n)}$$

$$v_0^{(n)} = v_{n-\frac{1}{2}} \wedge v_{n-3/2} \wedge \cdots$$

Back to dimers

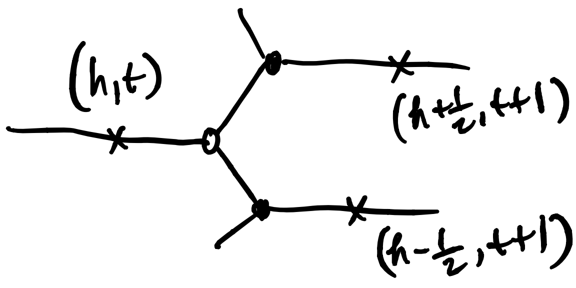
(on a hexagonal lattice)



$b(h,t) = (h, t-\frac{1}{2})$, $w(h,t) = (h, t+\frac{1}{2})$

Kasteleyn matrix: (b, w) with $b \sim w$

$$K(h,t) = (h,t) - (h+\tfrac{1}{2}, t+1) +$$
$$+ x_{h,t}(h-\tfrac{1}{2}, t+1)$$



Place fermion $a^*_{h,t}$ at $b(h,t)$

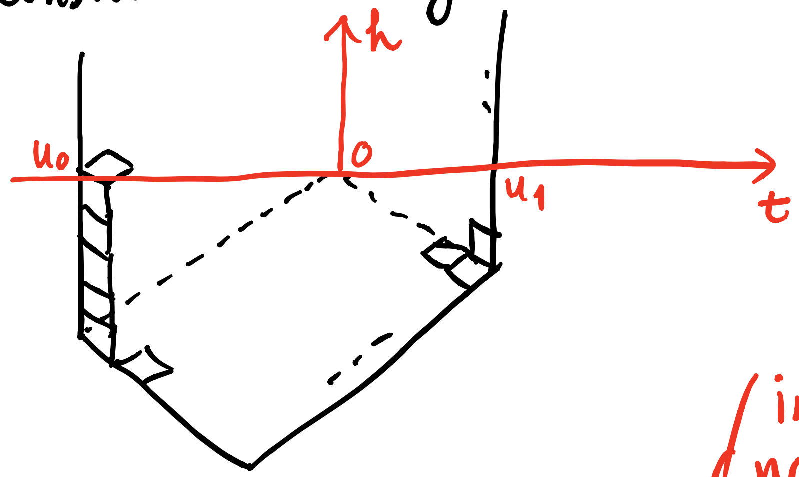
$a_{h,t}$ at $w(h,t)$

$$a^* K a = \sum_{h,t} a^*_{h,t} a_{h,t} - \sum_{h,t} a^*_{h+\frac{1}{2},t+1} a_{h,t}$$

$$+ \sum_{h,t} a^*_{h-\frac{1}{2},t+1} a_{h,t} x_{h,t} =$$

$$= \sum_t \left(a^*_t a_t + a_t V a^*_{t+1} - a_t V^{-1} X_t a^*_{t+1}\right)$$

Consider boundary conditions:



$\mathrm{Prob}(\pi) \propto$

$\propto \prod_t q_t^{|\pi(t)|}$

Assume that $x_{h,t} = x_t$. (in previous notations $q_{h,t} = q_t$)

Thm. $Z = \det(K) = \int e^{a^* K a} \, da^* da =$

$$= \left(\Gamma_-(x_{-\frac{1}{2}}) \cdots \Gamma_-(x_{u_0+\frac{1}{2}}) \Gamma_+(x_{\frac{1}{2}}) \cdots \Gamma_+(x_{u_1-\frac{1}{2}}) v_0^{(0)}, v_0^{(0)} \right)$$

Proof (outline)

$$\int \cdots \exp\left(a^*_{t-1} a_{t-1}\right) \exp\left(a_{t-1}\left(V - V^{-1} X_t\right) a^*_t\right)$$
$$\exp\left(a^*_t a_t\right) \exp\left(a_t\left(V - V^{-1} X_t\right) a^*_{t+1}\right) \cdots$$

$$= \cdots \widetilde{\left(V - V^{-1} X_{t+1}\right)}^{-1} \widetilde{\left(V - V^{-1} X_t\right)}^{-1} \cdots$$

$\Gamma_+(x_t)$ or $\Gamma_-(x_t)$ depending on t

$\tilde{A}$ is $A : V \circlearrowleft$ lifted to $\Lambda^{\frac{\infty}{2}} V$

$$V = \bigoplus_{h \in \mathbb{Z} + \frac{1}{2}} \mathbb{C} v_h \,,$$

One should take into account boundary conditions etc. ∎

Remark This can also be proven directly, combinatorially, bypassing Kasteleyn formula

From the algebra of vertex operators:

$$Z = \prod_{m=\frac{1}{2}}^{u_1-\frac{1}{2}} \prod_{m'=u_0+\frac{1}{2}}^{-\frac{1}{2}} (1 - x^-_{m'} x^+_m)^{-1}$$

Thm (Okounkov, R., 2005)

$$\langle \sigma(h_1,t_1) \cdots \sigma(h_k t_k) \rangle =$$

$$= \det \left(K((t_i,h_i),(t_j h_j)) \right)_{1 \le i,j \le k}$$

$$K((t_1 h_1)(t_2 h_2)) =$$

$$= \frac{1}{(2\pi i)^2} \int_{|z|<R(t_1)} \int_{|w|>\tilde{R}(t_2)} \frac{\Phi_-(z,t_1)\,\Phi_+(w,t_2)}{\Phi_+(z,t_1)\,\Phi_-(w,t_2)} \cdot$$

$$\cdot \frac{1}{z-w} z^{-h_1-B(t_1)-1/2} w^{h_2+B(t_2)-1/2} dz\, dw$$

$|w|<|z|$, $t_1 \geq t_2$

$|w|>|z|$, $t_1 < t_2$

$$R(t) = \min_{m>t} \left((x^+_m)^{-1} \right)$$

$$\tilde{R}(t) = \max_{m<t} (x^-_m)$$

$$\Phi_+(z,t) = \prod_{m>\max(t,\frac{1}{2})} (1 - z x_m^+)$$

$$B(t) = \frac{|t|}{2} - \frac{|t-u_0|}{2}$$

$$\Phi_-(z,t) = \prod_{m<\min(t,-\frac{1}{2})} (1 - z^{-1} x_m^-)$$

Thermodynamic limit with scaling

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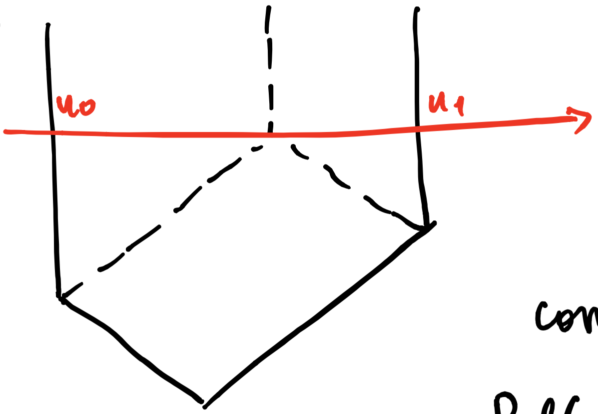


$$\left.\begin{aligned} x_m^+ &= a\, q^m \\ x_m^- &= a^{-1} q^{-m} \end{aligned}\right\} \text{assume}$$

correspond to

$$\mathrm{Prob}(\pi) \propto q^{|\pi|},$$

Consider the limit:

$q = e^{-\varepsilon}$, $u_1 = \frac{1}{\varepsilon} U_1$, $u_0 = \frac{1}{\varepsilon} U_0$, U_1, U_0 – fixed

$\varepsilon \to 0$

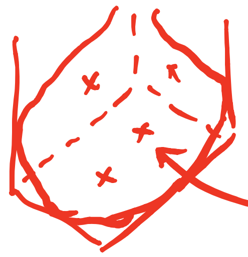
$$Z = \prod_{\substack{u_0<n<0 \\ 0<m<u_1}} (1 - x_m^+ x_n^-)^{-1} = \prod_{\substack{u_0<n<0 \\ 0<m<u_1}} (1 - q^{m-n})^{-1}$$

$$\langle |\pi| \rangle = q \frac{\partial}{\partial q} \ln Z$$

$$\left\{\begin{aligned} \ln Z &= \frac{1}{\varepsilon^2}\int_0^{U_1}\int_{U_0}^{0} \ln\left(1-e^{-s+t}\right) ds\, dt + \dots \quad \leftarrow \text{2D partition fncn} \\ \langle |\pi| \rangle &= \frac{1}{\varepsilon^3}\int_0^{U_1}\int_{U_0}^{0} \frac{s-t}{1-e^{t-s}}\, ds\, dt + \dots \quad \leftarrow \text{3D volume} \end{aligned}\right.$$

② Asymptotic of correlation functions

$t_i = \frac{\tau_i}{\varepsilon}$, $h_i = \frac{x_i}{\varepsilon}$, $\varepsilon \to 0$, τ_i, x_i fixed



(τ_i, x_i) in the bulk

$$K\big((t_1,h_1)(t_2,h_2)\big) \to$$

$$\to \frac{1}{(2\pi i)^2} \int_{C_z}\int_{C_w} e^{\frac{S(z,\tau_1,x_1)-S(w,\tau_2,x_2)}{\varepsilon}} \frac{\sqrt{zw}}{z-w}\frac{dz}{z}\frac{dw}{w}$$

$$S(z,\tau,x) = -\left(x+\frac{\tau}{2}-U_0\right)\ln z +$$

$$+\ \mathrm{Li}_2(z e^{-U_0}) + \mathrm{Li}_2(z e^{-U_1}) - \mathrm{Li}_2(z) - \mathrm{Li}_2(z e^{-\tau})$$

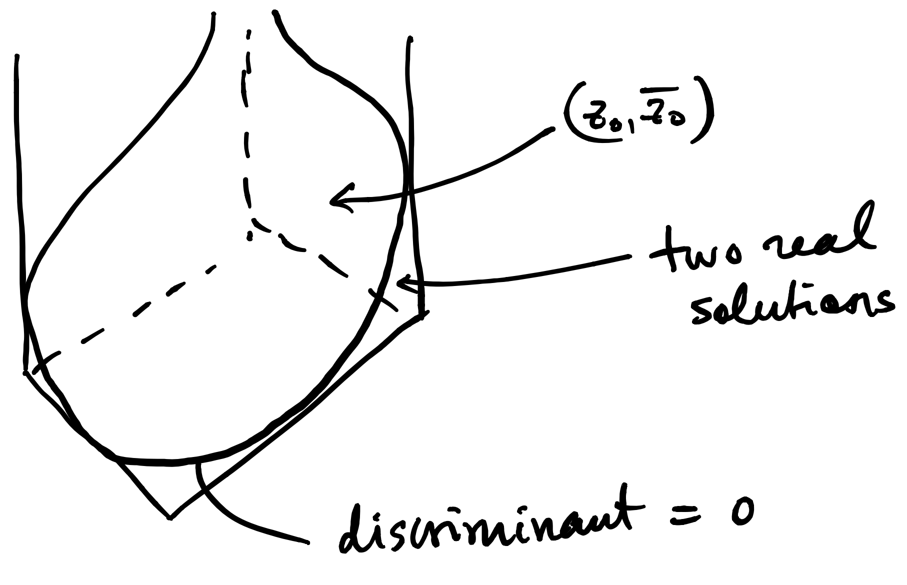
$$\mathrm{Li}_2(z) = \int_0^z \frac{\ln(1-t)}{t}\, dt$$

Evaluate by the steepest descent

Critical points of $S(z)$:

$$e^{\chi+\tau/2} = \frac{(1-ze^{-U_0})(1-ze^{-U_1})}{(1-z)(1-ze^{-\tau})}$$

Quadratic equation. Either 2 real solutions, or 2 complex conjugate or the discriminant is zero.



$$\boxed{\partial_x h_0(\tau, x) = \frac{1}{\pi} \arg(z_0)}$$

$$\langle \sigma_{(t,h)} \rangle = K((t,h)(t,h)) \to \varepsilon\, \partial_x h_0(\tau, x)$$

Steepest descent: (result)

$$K((t_1,h_1),(t_2,h_2)) = -\frac{\varepsilon}{2\pi}$$

$$\left(\frac{e^{\frac{S_1(z_1)-S_2(w_2)}{\varepsilon}}}{(z_1-w_2)\sqrt{-w_2 S_2''(w_2)}\sqrt{z_1 S_1''(z_1)}} - \frac{e^{\frac{S_1(z_1)-S_2(\bar{w}_2)}{\varepsilon}}}{(z_1-\bar{w}_2)\sqrt{-\bar{w}_2 S_2''(\bar{w}_2)}\sqrt{z_1 S_1''(z_1)}} + c.c. \right)$$

$$(1+o(1))$$

$$z_1 = z_0(x_1,\tau_1), \quad w_2 = z_0(x,\tau)$$

$z_0(x,\tau)$: inner part of the limit shape $\rightarrow H_+ = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$

From here:

$$K\big((t_1,h_1)(t_2,h_2)\big) = -\frac{\varepsilon}{2\pi}\, e^{\frac{1}{\varepsilon}\left(\mathrm{Re}\left(S(z_0(\tau_1,x_1))\right) - \mathrm{Re}\left(S(z_0(\tau_2,x_2))\right)\right)} \cdot$$

$$\left(\frac{\exp\left(\frac{i}{\varepsilon}\left(\mathrm{Im}\, S'(z_1) - \mathrm{Im}\, S(w_2)\right)\right)}{z_1 - w_2} + \frac{\exp\left(\frac{i}{\varepsilon}\left(\mathrm{Im}\, S'(z_1) + \mathrm{Im}\, S(w_2)\right)\right)}{z_1 - \bar{w}_2} \right.$$

$$\left. + \text{ c.c. } \right) \big(1 + o(1)\big) \qquad (*)$$

This suggests the convergence of Kasteleyn fermions to free Dirac fermions:

$$\frac{1}{\sqrt{\varepsilon}}\psi_{\vec{x}} = e^{\frac{1}{\varepsilon}\mathrm{Re}(S(z_0))} \left(\psi_+(z_0)\, e^{\frac{i}{\varepsilon}\mathrm{Im}(S(z_0))} + \psi_-(\bar{z}_0)\, e^{-\frac{i}{\varepsilon}\mathrm{Im}(S(z_0))} \right)\big(1+o(1)\big)$$

$$\frac{1}{\sqrt{\varepsilon}}\psi^*_{\vec{x}} = e^{\frac{1}{\varepsilon}\mathrm{Re}(S(z_0))} \left(\psi^*_+(z_0)\, e^{-\frac{i}{\varepsilon}\mathrm{Im}(S(z_0))} + \psi^*_-(\bar{z}_0)\, e^{\frac{i}{\varepsilon}\mathrm{Im}(S(z_0))} \right)\big(1+o(1)\big)$$

$$\mathbb{E}\left(\psi_{\pm}^{*}(z)\,\psi_{\pm}(w)\right) = \frac{1}{z-w},$$

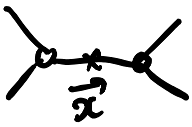
$$\mathbb{E}\left(\psi_{\pm}^{*}(z)\,\psi_{\mp}(w)\right) = \mathbb{E}(\psi^{*}\psi^{*}) = \mathbb{E}(\psi\psi) = 0$$

$\psi_{\pm}^{*}(z)$, $\psi_{\pm}(w)$ are spinors:

$$\psi_{\pm}^{*}(z) = \psi_{\pm}^{*}(w)\sqrt{\frac{\partial w}{\partial z}}$$

$$\psi_{\pm}(z) = \psi_{\pm}(w)\sqrt{\frac{\partial w}{\partial z}}$$

For dimer correlation functions we have



$$\left\langle \left(\sigma_{\vec{x}_1} - \langle \sigma_{\vec{x}_1}\rangle\right)\left(\sigma_{\vec{x}_2} - \langle \sigma_{\vec{x}_2}\rangle\right)\right\rangle$$

$$= K_{12}K_{21} =$$

$$= \frac{\varepsilon^2}{(2\pi)^2}\left[\frac{\frac{\partial z_1}{\partial x_1}\,\frac{\partial w_2}{\partial x_2}}{(z_1-w_2)^2} - \frac{\frac{\partial z_1}{\partial x_1}\,\frac{\partial \bar{w}_2}{\partial x_2}}{(z_1-\bar{w}_2)^2} + \right.$$

$$\left. + \text{c.c.}\right]\left(1+o(1)\right)$$

or :

$$\sigma_{\vec{x}} - \langle \sigma_{\vec{x}} \rangle = \varepsilon\, \partial_x \varphi(z_\delta(\tau, x)) + \cdots$$

$\varphi(z)$ - Gaussian free field on H_+

$$\langle \varphi(z)\, \varphi(w) \rangle = \frac{1}{2\pi} \ln \left| \frac{z-w}{z-\bar{w}} \right|$$

↑

Green's fncn of the Dirichlet problem on H_+.

Bose - Fermi corresponence :

$$\partial_x \varphi = :\tilde{\psi}(z,\bar{z})\, \tilde{\psi}(z,\bar{z}):$$

....