

Limit shapes in statistical mechanics

- Lect 1-2 :
- dimer models on graphs
 - combinatorial equivalences
 - Kasteleyn solution to dimer models
 - Mapping of Ising model to dimer models

- Lect 3-4 .
- Thermodynamic limit
 - The limit shape phenomenon
 - Analogy with coin tossing
 - The variational principle for the limit shape
 - Gaussian field theory

Dimer models on surface graphs

Recall:

$$Z_{\Gamma}^{\text{dimer}} = \sum_{\mathcal{D} \subset \Gamma} \prod_{e \in \mathcal{D}} w(e)$$

For a plane graph $\Gamma \subset \mathbb{R}^2$

(i) If Γ is bipartite

• bijection $\{ \mathcal{D} \subset \Gamma \} \leftrightarrow \{ \text{h. func on } \Gamma \}$

An example of height function $h_{\mathcal{D}}$ given

$$Z_{\Gamma}^{\text{dimer}} = \text{const} \sum_{h \in \mathcal{H}_{\Gamma}} \prod_f q_f^{h(f)}, \quad q_f = \prod_{e \in f} w(e)^{\varepsilon(f,e)}$$

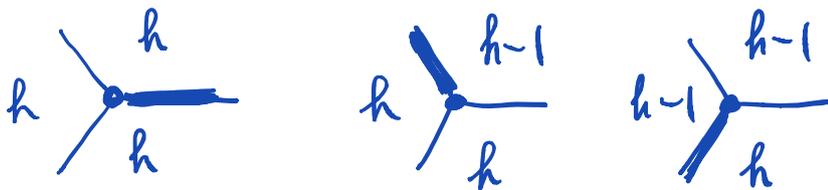
$$\text{Prob}(h) = \frac{\prod_f q_f^{h(f)}}{\sum_{h \in \mathcal{H}_{\Gamma}} \prod_f q_f^{h(f)}}$$

\mathcal{H}_Γ = the space of h.fncns, $h|_{\partial\Gamma}$ = determined by Γ

Note: Let $h = \tilde{h} + h^{(0)}$, $\mathcal{H}_\Gamma = \tilde{\mathcal{H}}_\Gamma + h^{(0)}$

Probabilities do not change $\mathbb{Z} \mapsto \text{const. } \mathbb{Z}$

An example:



and $h(f_0) = 0$

h/w. prove that this is an example.

(ii) Choose a Kasteleyn orientation of Γ (if \exists , unique up to an equivalence)

$$\mathbb{Z}_\Gamma^{\text{dimer}} = |\text{Pf}(A_\Gamma^K)|$$

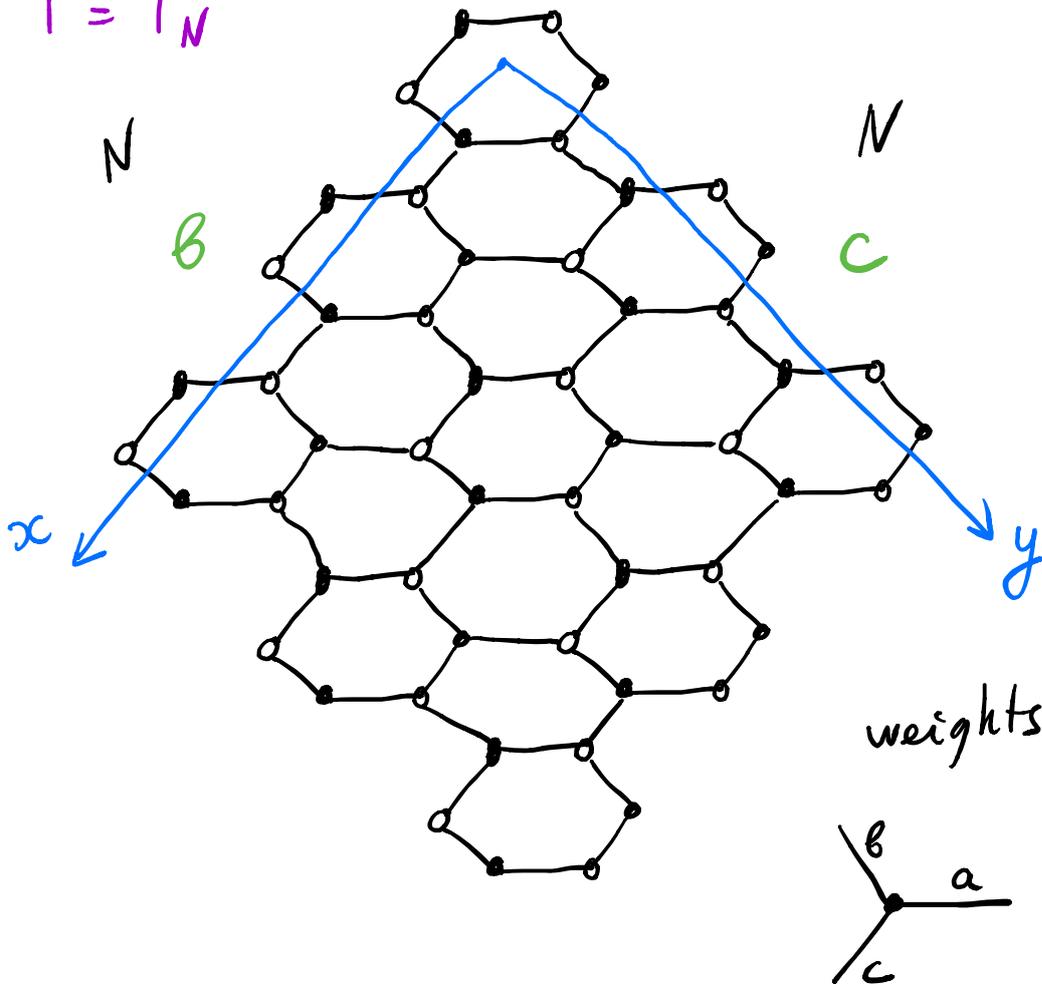
Surface graphs $\Gamma \subset \Sigma_g$ (Torus)

Assume that all faces of Γ contractible

(i) Thm. # (equivalence classes of K-orientations) = 2^{2g}

Now $\Sigma_g = T$, $g=1$ ($g>1$ tomorrow)

$\Gamma = T_N$



Nonequivalent Kasteleyn orientations:



K_{01} : \nearrow and \rightarrow replace by \nwarrow and \leftarrow on c cycle

K_{10} : \nearrow and \rightarrow replace by \nwarrow and \leftarrow on b cycle

K_{11} : \nearrow and \rightarrow replace by \nwarrow and \leftarrow on b & c cycles

Thm (Kasteleyn)

$$Z_{T_N}^{\text{dimur}} = \frac{1}{2} (Z_{00} + Z_{01} + Z_{10} - Z_{11})$$

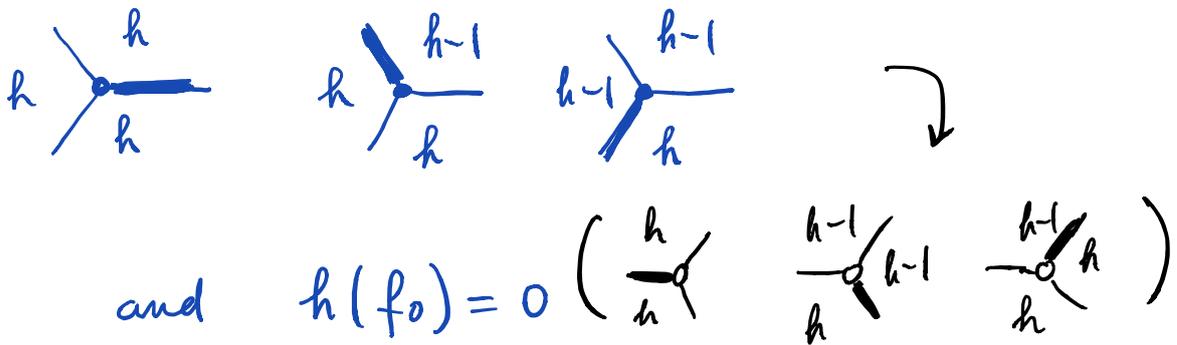
$$Z_{\alpha\beta} = \det(A^{K_{\alpha\beta}})$$

$$Z_{00} = \prod_{z^N = w^N = 1} (a + z\beta + wc), \quad Z_{01} = \prod_{z^N = -w^N = 1} (a + z\beta + wc)$$

$$Z_{10} = \prod_{-z^N = w^N = 1} (a + z\beta + wc), \quad Z_{11} = \prod_{z^N = w^N = -1} (a + z\beta + wc)$$

Proof h/w

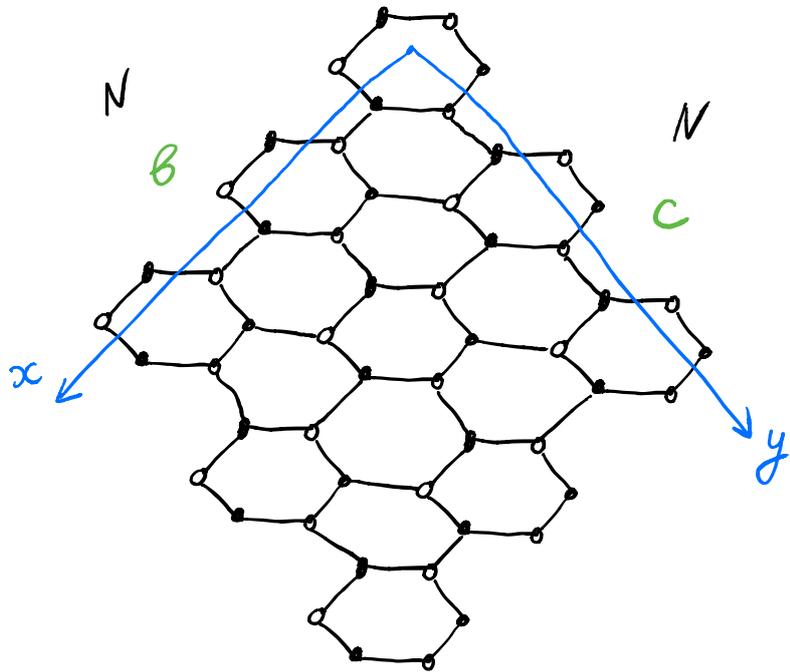
(ii) Choose the height function defined above:



Define:

$h_x =$ the gain of the h. fncn along c

$h_y =$ the gain of the h. fncn along b

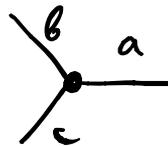


$N = \#$ Black (or white) vertices

Define

(a) $N_{h_x, h_y} = \#$ of dimer configurations with h. function gain h_x, h_y

h/w: prove that N_{h_x, h_y} is the number of dimer configurations with $N_a = N(N - h_x - h_y)$
 $N_b = Nh_x$, $N_c = Nh_y$, $N_a, N_b, N_c =$
 $= \#$ of a, b, c edges



(b) $Z_{h_x, h_y}^{\text{dimer}} = \sum_{\substack{\text{DCT}_N \\ \text{fixed } h_x, h_y}} \prod_{e \in \mathcal{D}} w(e)$

(c) The generating function for $Z_{h_x, h_y}^{\text{dimer}}$

$$Z_{TN}^{\text{dimer}}(H, V) = \sum_{h_x, h_y} Z_{h_x, h_y}^{\text{dimer}} e^{NHh_x} e^{NVh_y}$$

Thm 1)
$$Z_{h_x, h_y}^{\text{dimer}} = N_{h_x h_y} a^{N_a} b^{N_b} c^{N_c}$$

2)
$$Z_{T_N}^{\text{dimer}}(H, V) = a^{N^2} \sum_{h_x, h_y} N_{h_x h_y} \left(\frac{b}{a} e^H\right)^{N_{h_x}} \left(\frac{c}{a} e^V\right)^{N_{h_y}}$$

Corollary

$$\sum_{h_x, h_y} N_{h_x h_y} e^{NH_{h_x}} e^{VN_{h_y}} =$$

$$= \frac{1}{2} \sum_{\alpha, \beta=0,1} Z_{\alpha\beta}(H, V)$$

$$Z_{\alpha\beta}(H, V) = \prod_{\substack{z^N = (-1)^\alpha \\ w^V = (-1)^\beta}} (1 + ze^H + we^V)$$

(iii) $N \rightarrow \infty$ limit, $\exists a=b=c=1$

Thm As $N \rightarrow \infty$

$$(a) \quad \sum_{T_N} Z(H, V) = \exp(N^2 R(H, V) + o(N^2))$$

$R(H, V)$ = the free energy per site

$$R(H, V) = \left(\frac{1}{2\pi i} \right)^2 \int_{|z|=e^H} \int_{|w|=e^V} \ln(1+z+w) \frac{dz}{z} \frac{dw}{w}$$

$$(b) \quad N_{h_x h_y} = \exp(-N^2 \sigma(s, t) + o(N^2))$$

assuming $h_x = Ns$, $h_y = Nt$.

$$\sigma(s, t) = \max_{H, V} (Hs + Vt - R(H, V))$$

$$\sigma(s, t) = -\frac{1}{\pi} (L(\pi s) + L(\pi t) + L(\pi(1-s-t)))$$

$$L(\theta) = -\int_0^{\theta} \ln(2 \sin t) dt$$

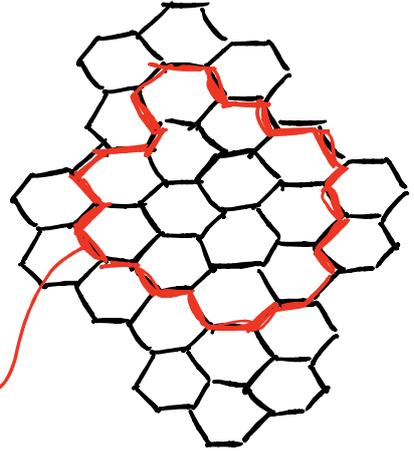
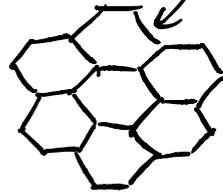
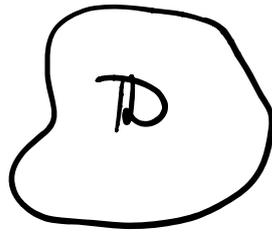
The thermodynamic limit

Convergent sequences of lattice domains

To be specific, consider 6-gon lattice ε

$$\varphi_\varepsilon : H \hookrightarrow \mathbb{R}^2$$

$$\mathbb{D} \subset \mathbb{R}^2$$



Fix $\{\varepsilon_n \rightarrow 0\}$

and a sequence of D_n

domains $D_n \subset \varphi_{\varepsilon_n}(\mathbb{Z}^2)$ s.t.

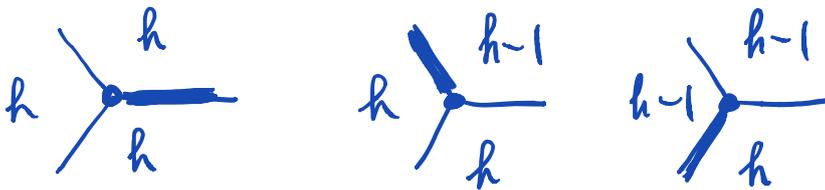
$$\text{dist}(D_n, D) \rightarrow 0, n \rightarrow \infty$$

We can say $|D_n| \rightarrow \infty$, $D_n \rightarrow D$
as $n \rightarrow \infty$

Continuous height functions

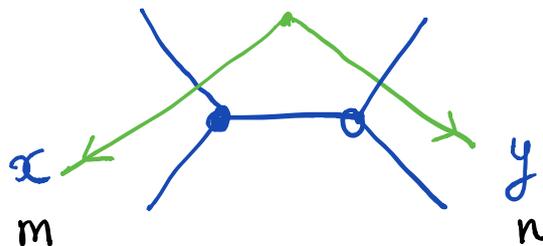
Assume we have a sequence of lattice domains converging to $\mathbb{D} \subset \mathbb{R}^2$

For each \mathbb{D}_n we have the space of height functions $H_{\mathbb{D}_n}$:



and $h(f_0) = 0$

Choose Euclidean coordinates as



For a height function we have:

$$0 \leq h(m+N, n) - h(m, n) \leq N$$

$$0 \leq h(m, n+N) - h(m, n) \leq N$$

If we want $\varepsilon h(\vec{m}) \rightarrow \tilde{h}(\varepsilon \vec{m})$
we can have only functions
with

$$0 \leq \tilde{h}(x+t, y) - \tilde{h}(x, y) \leq t$$

$$0 \leq \tilde{h}(x, y+t) - \tilde{h}(x, y) \leq t$$

Lipshitz conditions on $\tilde{h} : \mathbb{D} \rightarrow \mathbb{R}$

2) $h|_{\partial \mathbb{D}_n}$ is determined by \mathbb{D}_n
and by normalization $h(f_0) = 0$.

To define the space of continuous height functions on \mathbb{D} , we require that as $n \rightarrow \infty$

$$\varepsilon h|_{\partial \mathbb{D}_n} \rightarrow \chi : \partial \mathbb{D} \rightarrow \mathbb{R}$$

Naturally, χ should satisfy appropriate Lipschitz conditions (above)

Fact: for each χ satisfying these conditions $\exists \mathbb{D}_n \rightarrow \mathbb{D}$

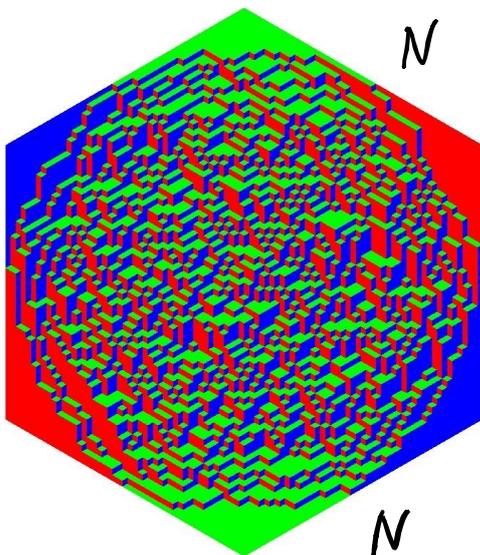
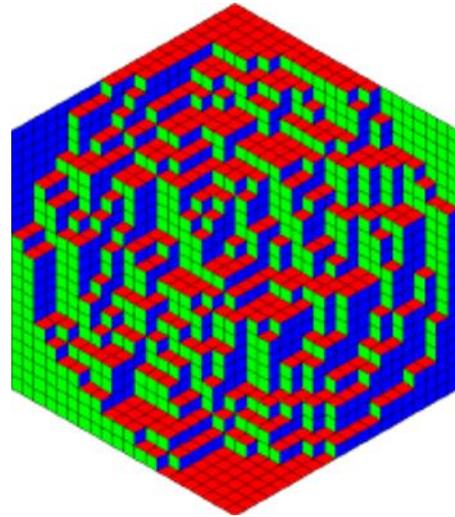
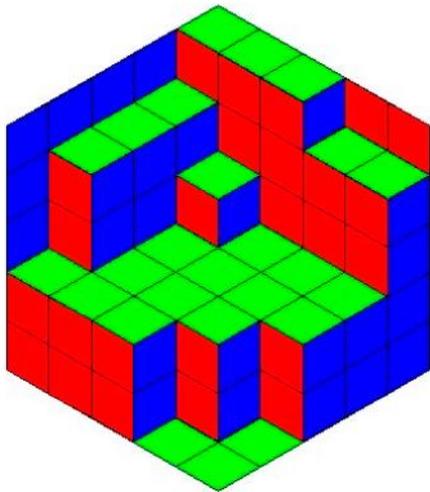
Def. The space of continuous height functions $\mathcal{H}_{\mathbb{D}, \chi} = \{h: \mathbb{D} \rightarrow \mathbb{R}, h|_{\partial \mathbb{D}} = \chi, h(x_0) = 0, \text{ Lipschitz conditions} \}$

The main question about the thermodynamic limit for dimer models on bipartite lattices:

Given a sequence $D_n \rightarrow D$ corresponding to the boundary value φ , describe the asymptotic of the probab. measure in this limit

- Find the asymptotic of Z (weighted number of states)
- Find the asymptotic of correlation functions

Examples:



Random states
for uniform
measure $w(e)=1$

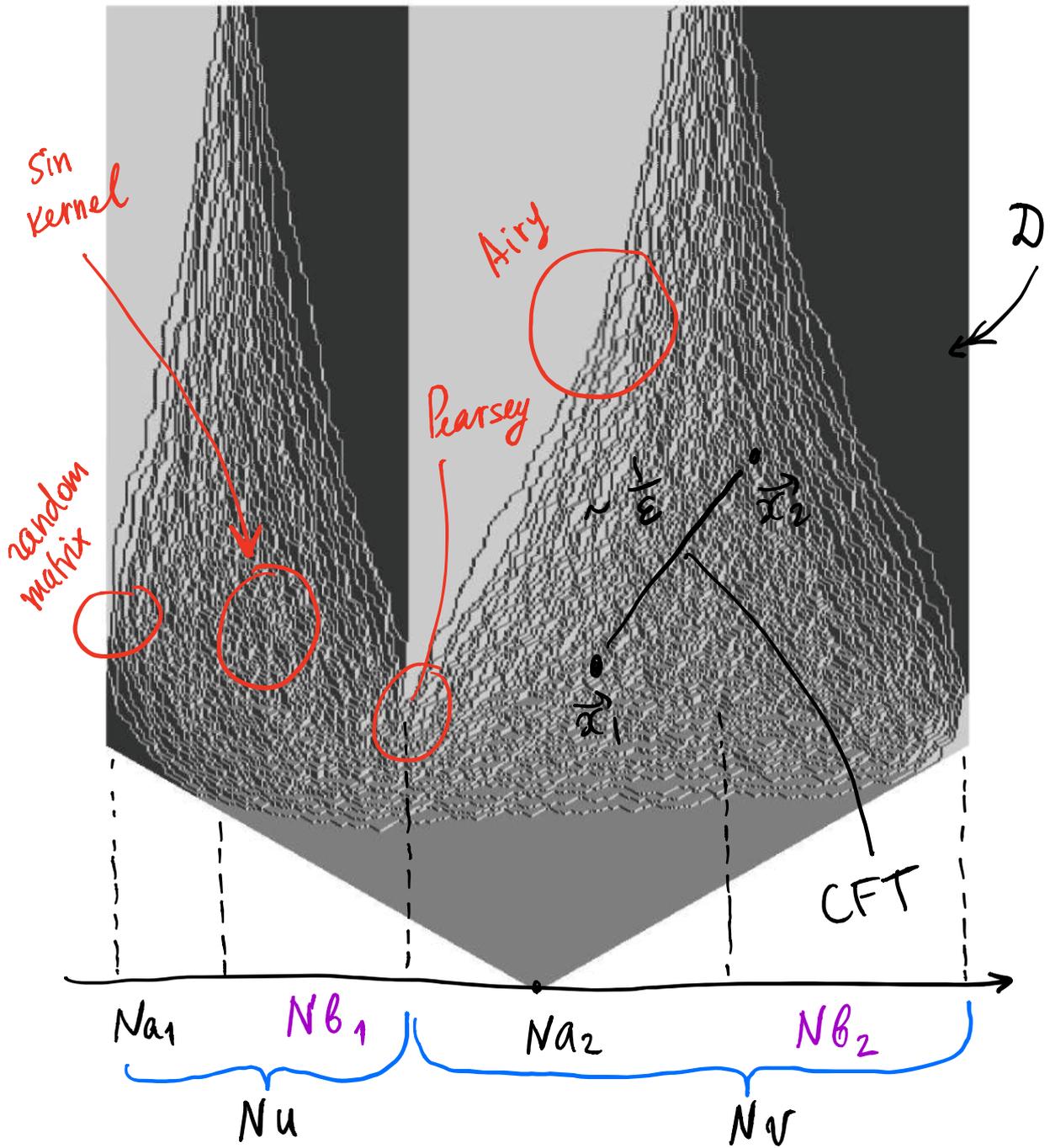
$$\text{Prob}(h) = \frac{1}{|\text{states}|}$$

$$h \in \mathcal{H} = \{\text{linear functions}\}_{\text{critical}}$$

Asymptotic of correlation functions
depends on the scale of distances

$$\text{Prob}(h) \propto q^{\text{vol}(h)}, \quad h|_{\partial D} = \{ \text{linear functions} \}_{\text{critical}}$$

$$u+v = a_1 + a_2 + b_1 + b_2, \quad N = \frac{1}{\epsilon}, \quad q = e^{-\epsilon}$$



On these pictures we see:

$$h(\vec{n}) \rightarrow \frac{1}{\varepsilon} h_0(\vec{x}) + \varphi(\vec{x}), \quad \varepsilon \rightarrow 0$$

↑ ↑ ↑
random deterministic random

convergence in probability

$\vec{x} = \varepsilon \vec{n}$ is finite $\vec{x} \in \mathcal{D}$

$h_0(\vec{x})$ - continuous height function
with p.w. linear boundary conditions
(for these particular domains)

$\varphi(\vec{x})$ - continuous Gaussian field

h_0 = the limit shape.

It is determined by the variational principle.

The variance of $\varphi(\vec{x})$ is also determined by the same principle.

Digression

Coin tossing: N tossings

2^N states $\{ \underbrace{\bullet \ 0 \ 0 \ \bullet \ \bullet \ \dots \ 0}_{N} \}$

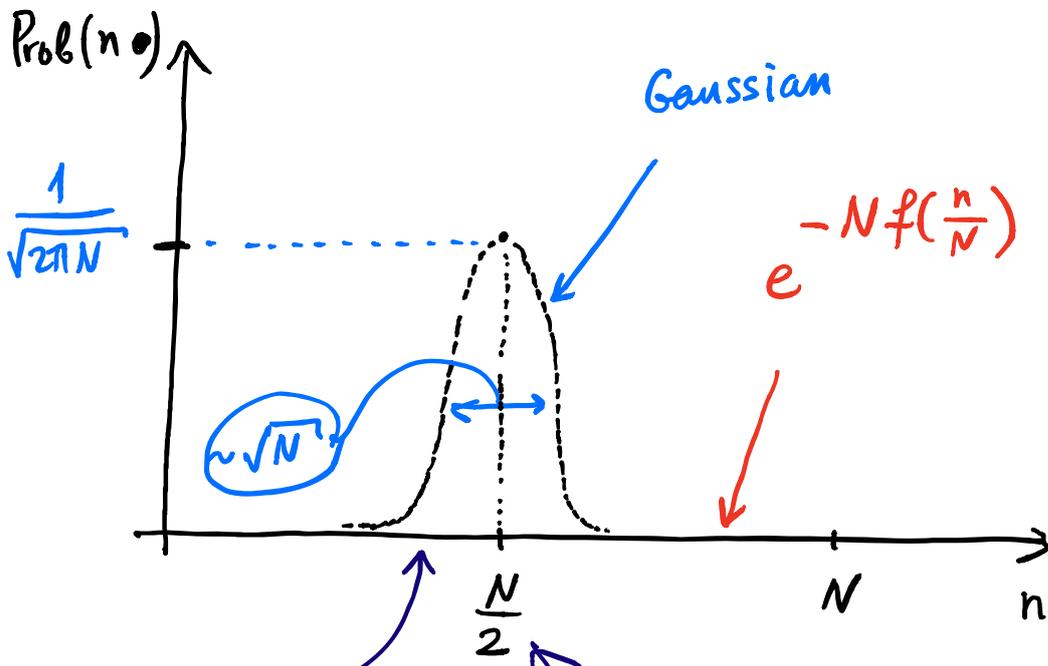
$$\text{Prob}(n \text{ of } \bullet) = \frac{1}{2^N} \binom{N}{n}$$

$N \rightarrow \infty$, $x = \frac{n}{N}$, Stirling formula:

$$\text{Prob}(n \text{ of } \bullet) = \begin{cases} \exp(-N f(x)), & x \neq \frac{1}{2} \\ \frac{1}{\sqrt{2\pi N}} e^{-\frac{y^2}{2}}, & x = \frac{1}{2} \end{cases}$$
$$n = \frac{N}{2} + \sqrt{N} y$$

$$f(x) = x \ln x + (1-x) \ln(1-x) + \ln 2$$

Large deviation rate function



fluctuations

The limit shape

$$n = \frac{N}{2} + y\sqrt{N}$$

↑
random

random

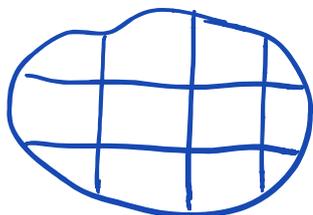
deterministic

The variational principle

$$\mathcal{Z}_{\mathbb{D}_n}^{\text{dimer}} = \sum_{h \in \mathcal{H}_{\mathbb{D}_n}} \prod_f q_f^{h(f)}$$

assume $q_f = \exp(\epsilon_n \lambda(\vec{x}))$,

$$\mathcal{Z}_{\mathbb{D}_n}^{\text{dimer}} \approx \sum_{h \text{ on } i} \prod_i \mathcal{Z}_{\mathbb{D}_n^{(i)}} \approx$$



$$\approx \sum_{h \text{ on } \text{boundaries}} \exp(N(\mathbb{D}^{(i)}) \sigma(h_x^{(i)}, h_y^{(i)})) \approx$$

$$\approx \exp(N(\mathbb{D}_n) \iint_{\mathbb{D}} \sigma(\partial_x h_0, \partial_y h_0) dx dy)$$

h_0 is the minimizer of

$$S[h] = \iint_{\mathbb{D}} \sigma(\partial_x h, \partial_y h) dx dy$$

$$h \in \mathcal{H}_\chi, \text{ i.e. } h|_{\partial\mathbb{D}} = \chi$$

$$0 \leq \partial_x h, \partial_y h \leq 1$$

Dispersion for fluctuations:

$$\delta^2 S[h] = \iint_{\mathbb{D}} \left(\partial_x h^2 \partial_1^2 \sigma(\vec{\nabla} h_0) + 2 \partial_x h \partial_y h \cdot \partial_1 \partial_2 \sigma(\vec{\nabla} h_0) + \partial_y h^2 \partial_2^2 \sigma(\vec{\nabla} h_0) \right) dx dy$$

Next time