Limit shapes in statistical mechanics (integrable models)

Plan:

1. Dimer models
   - Combinatorics, Pfaffian solution
   - Thermodynamic limit, limit shapes, variational principle, correlation functions

2. The 6-vertex model
   - The Yang-Baxter equation
   - Commuting transfer-matrices & their spectrum
   - The limit shape & the variational principle
   - Fluctuations

Equilibrium statistical mechanics

- X - finite set, space of states
- $E : X \to \mathbb{R}$ the energy function $E(x)$
- The Boltzmann probability distribution on X (assuming that X is in the thermodynamic equilibrium, isolated)

$$\boxed{\mathrm{Prob}(x) \propto \exp\left(-\frac{E(x)}{kT}\right)}$$

k = Boltzmann constant
T = temperature

The normalizing factor: <u>the partition fncn</u>

$$\boxed{Z = \sum_{x \in X} \exp\left(-\frac{E(x)}{kT}\right)}$$

$$\boxed{\mathrm{Prob}(x) = \frac{\exp\left(-\frac{E(x)}{kT}\right)}{Z}}$$

normalized probability

<u>Observables</u>: fncns on X

if $f : X \to \mathbb{R}$,

$$\mathbb{E}[f] = \sum_x f(x)\,\mathrm{Prob}(x),$$

or equivalently (physics notations)

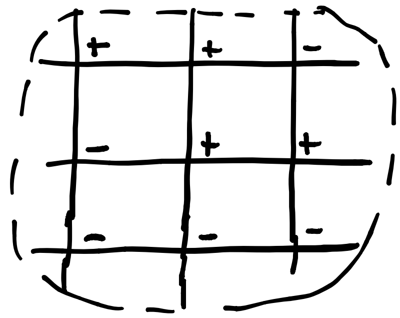
$$\langle f \rangle = \frac{\sum_x f(x)\, e^{-\frac{E(x)}{kT}}}{Z},$$

For finite X finding Z and $\langle f \rangle$ are combinatorial problems. Some time very nice formulae.

Hard analytical problem: "$|X| \to \infty$"

Ising model on a graph Γ.

Γ



$\mathcal{E}x$: $\Gamma \subset \mathbb{Z}^2$

assume connected,
simply connected

$X = \{+,-\}^{V(\Gamma)}$

$V(\Gamma)$ = vertices of Γ,

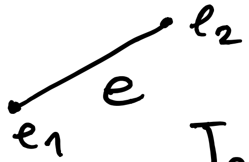
$E(\Gamma)$ = edges of Γ,

$|X| = 2^{|V(\Gamma)|}$

A state $\sigma = \{\sigma_v\}_{v \in V(\Gamma)}$

Energy function

$$E(\sigma) = -\sum_{e \in E(\Gamma)} J_e \, \sigma_{e_1} \sigma_{e_2} - \mu H \sum_v \sigma_v$$



symmetric

$\sigma_v = \pm 1$,

$J_e > 0$ exchange energy

$$\mathrm{Prob}(\sigma) \propto \exp\left(-\frac{E(\sigma)}{kT}\right),$$

1) $T \to 0$ most probable is the state with min energy (when $H=0$, either $\sigma_v = +1$ for all v, or $\sigma_v = -1$, for all v).

2) $T \to \infty$ in this limit

$$\mathrm{Prob}(\sigma) \to \frac{1}{\#(\text{states})}$$

the uniform distribution

Local observables

$\sigma_{v_1} \cdots \sigma_{v_n}$ (fncn on X with values ± 1)

Local correlation fncn

$\langle \sigma_{v_1} \cdots \sigma_{v_n} \rangle$

Questions natural to statistical mechanics:

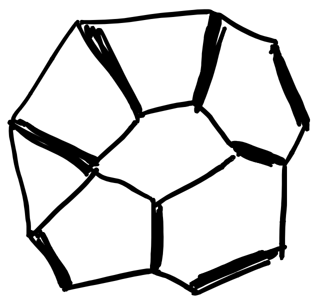
$$|\Gamma| \to \infty$$

- asymptotic of probability measure
- partition functions
- correlation functions (local, ...)

Will focus on this later

Dimers on a graph Γ

<u>Def</u>. Dimer configuration on Γ is a perfect matching on vertices connected by edges.



- occupied edges can not overlap on vertices
- all vertices should be covered

Energies of edges in Γ

$$E : E(\Gamma) \to \mathbb{R}, \quad e \mapsto E(e)$$

Energy of a dimer configuration:

$$E(D) = \sum_{e \subset D} E(e)$$

Probability of a dimer configuration:

$$\mathrm{Prob}(D) = \frac{\exp\left(-\frac{E(D)}{kT}\right)}{Z} = \frac{\prod_{e\in D} e^{-\frac{E(e)}{kT}}}{Z}$$

$$Z = \sum_{D} \exp\left(-\frac{E(D)}{kT}\right)$$

Local weights:

$$w(e) = e^{-\frac{E(e)}{kT}}$$

In terms of local weights:

$$\mathrm{Prob}(D) = \frac{\prod_{e\subset D} w(e)}{\sum_{D} \prod_{e\subset D} w(e)},$$

Local observables (conditional probabilities)

$$\langle \sigma_{e_1} \cdots \sigma_{e_n} \rangle = \mathrm{Prob}(e_i \subset D)$$

or , consider functions

$$\sigma_e(D) = \begin{cases} 1, & e \subset D \\ 0, & e \not\subset D \end{cases}$$

then

$$\langle \sigma_{e_1} \cdots \sigma_{e_n} \rangle = \sum_{D \subset \Gamma} \sigma_{e_1}(D) \cdots \sigma_{e_n}(D) \, \mathrm{Prob}(D) =$$

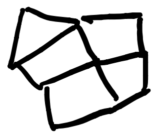
$$= \mathbb{E}(\sigma_{e_1} \cdots \sigma_{e_n})$$

Combinatorial equivalences

① Dimers $\longleftrightarrow$ tilings

Assume $\Gamma \subset \mathbb{R}^2$ is a <u>plane graph</u>

- $\Gamma \subset \mathbb{R}^2$ defines 2-dimensional cell complex



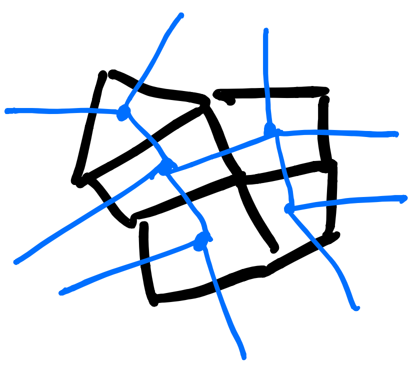
- vertices — 0-cells
- edges — 1-cells
- regions (faces) — 2-cells

- Dual cell complex $\Gamma^\vee$:

0-cells = "centers" of 2-cells of Γ

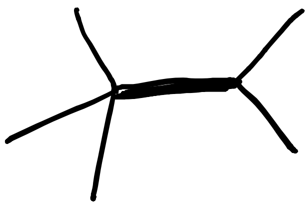
1-cells = "centers" of 1-cells of Γ

2-cells = "centers" of 0-cells of Γ

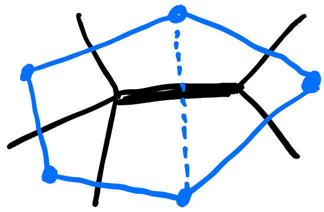


$\Gamma^\vee$ – dual cell complex to Γ

- Dimer on Γ

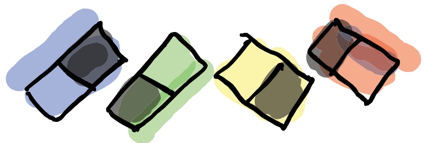


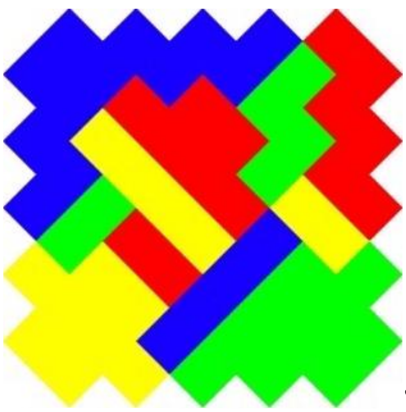
Pair of 2-cells on $\Gamma^\vee$ sharing an edge

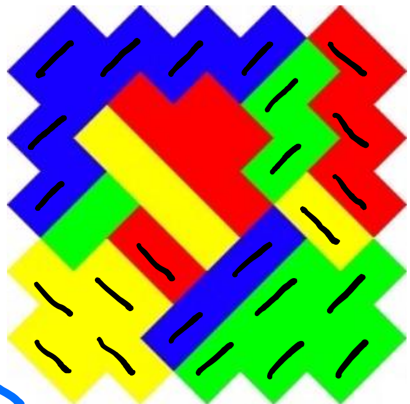


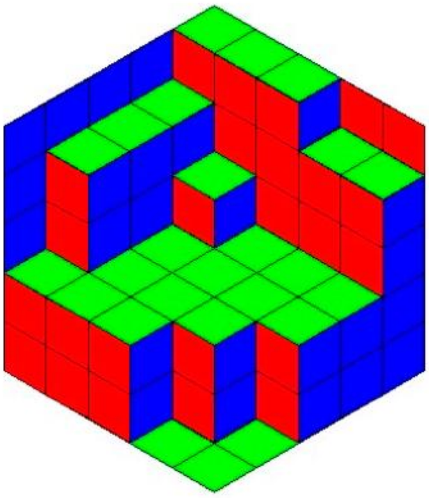
- This is a global bijection

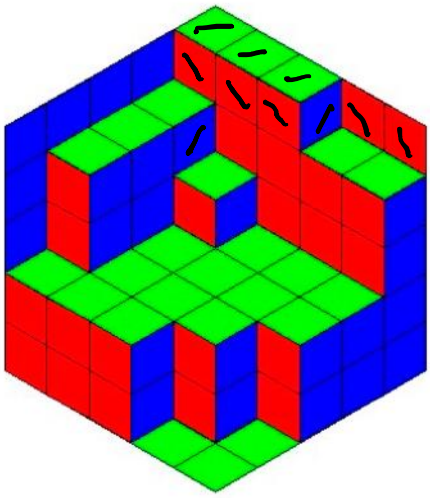
$$\left(\text{Dimers on } \Gamma\right) \longleftrightarrow \left(\begin{array}{c}\text{tilings of } \Gamma^\vee \text{ by} \\ \text{pairs of 2-cells}\end{array}\right)$$

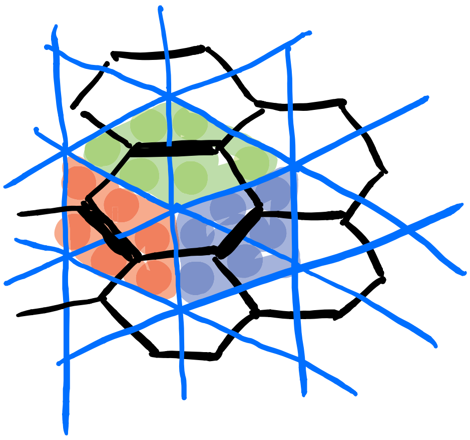
Color = bipartite structure 











Same for $\Gamma \subset \Sigma_g$

(dimers on Γ) $\leftrightarrow$

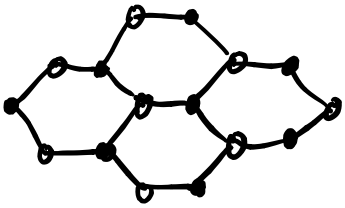
$\leftrightarrow$ (tilings of $\Gamma^\vee$ by double tiles)

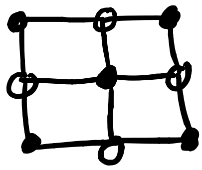
Bipartite graphs

$$V(\Gamma) = V_\bullet(\Gamma) \sqcup V_\circ(\Gamma)$$

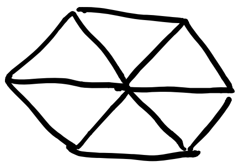
such that no edges between black & black and white & white

Ex.



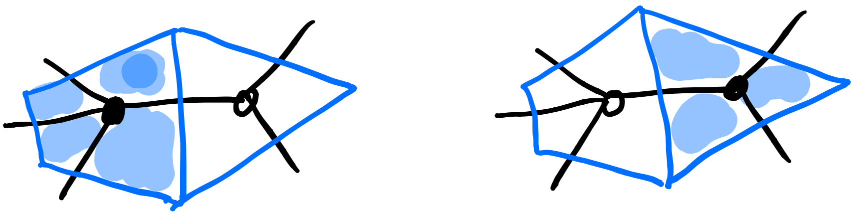


Non example

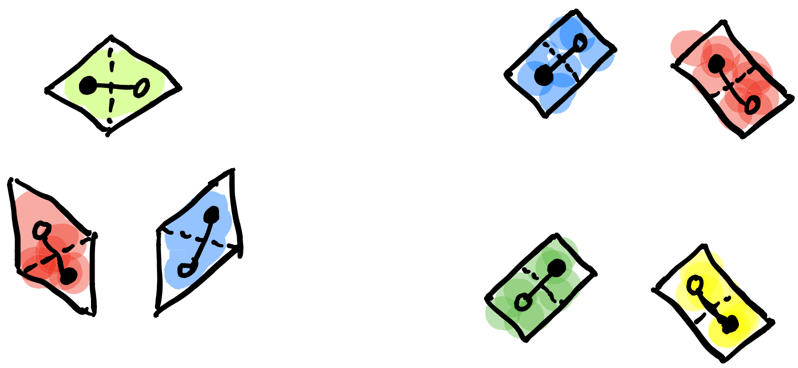


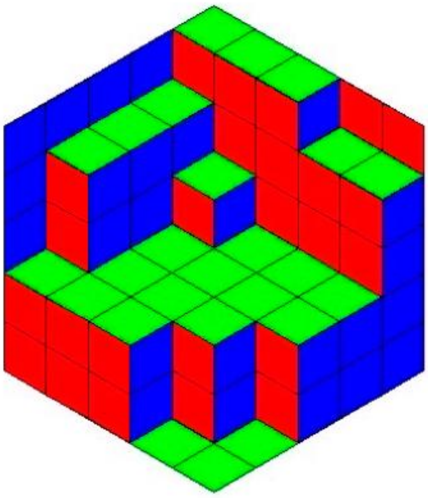
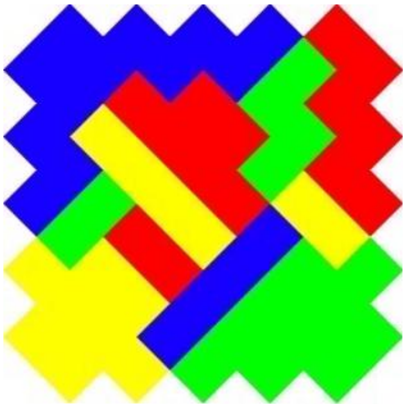
no bipartite structure on a triangular lattice

On a dual cell complex to a bipartite graph:



Two "colors" of the same tile



Height function

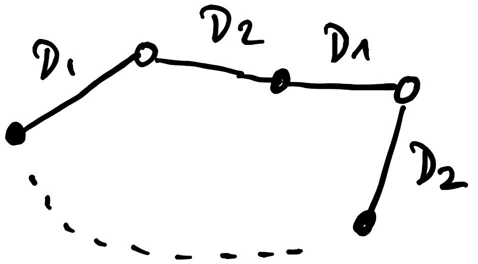
Dimers $\longleftrightarrow$ discrete surfaces

From now on assume

i) the graph is plane $\Gamma \subset \mathbb{R}^2$

ii) Γ is bipartite

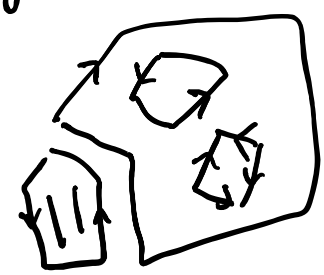
Let $D_1, D_2 \subset \Gamma$ be a pair of dimer coverings of Γ



cycle on Γ formed by dimers from D_1 & D_2

Orient edges: 

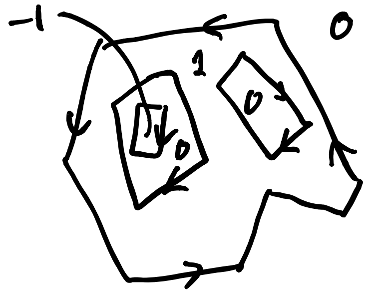
The result is a system of composition cycles:



- oriented
- non intersecting,
- nonself intersecting
- covering all vertices

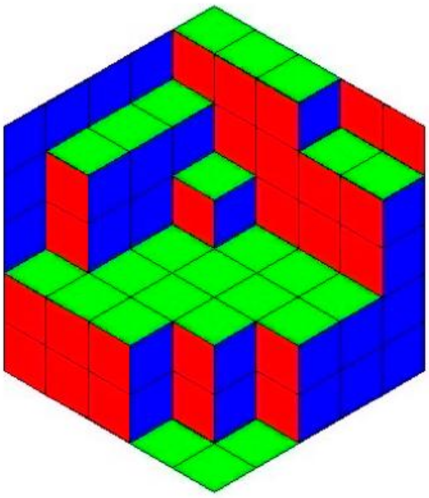
Remove cycles of length 2

The resulting system of oriented cycles = level curves of a function on faces = height function $h_{D_1 D_2}$



$h_{D,D_0}(f) = 0$

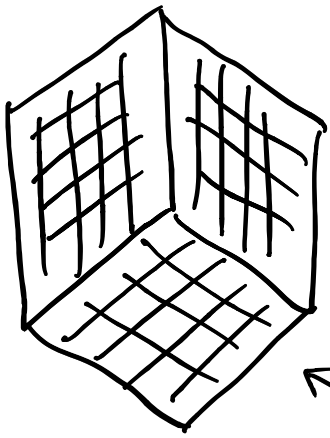
if f is the outer face



$h_{D,D_0}(f)$ is defined on vertices of this picatre =

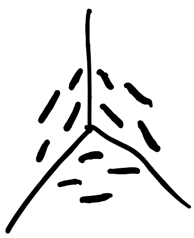
= the distance to the "wall" along $(1,1,1)$ direction

D_0 = "wall"



tiling

$\longleftrightarrow$

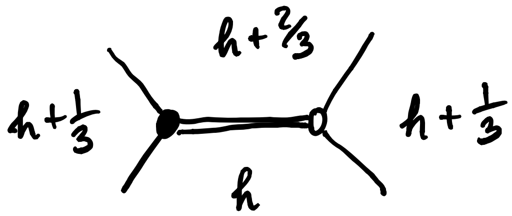


dimers

Thm. $h_{D_1D_2} + h_{D_2D_3} = h_{D_1D_3}$

$h_{D_1D_2}$ = the relative height fncn.

For hexagonal lattice: the absolute height fncn



h/w: prove that

(i) $h_D\big|_{\partial\Gamma}$ does not depend on D
$\partial\Gamma$ ← boundary faces

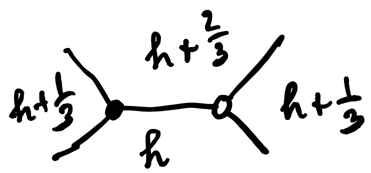
(ii) $h_{D_1D_2} = h_{D_1} - h_{D_2}$

(iii) Construct h_D for the square grid

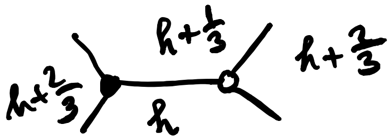
Define the space of height functions

as

$$\mathcal{H}_\Gamma = \{ h : \text{faces}(\Gamma) \to \mathbb{Z} \}$$

- 

- normalization $h(f_0) = 0$

- 

<u>Proposition</u> Dimers on $\Gamma \simeq$ height fncns from $\mathcal{H}_\Gamma$

<u>Proof</u>: h/w

<u>Thm</u> If $h_1, h_2 \in \mathcal{H}_\Gamma$, then

$$h_1|_{\partial\Gamma} = h_2|_{\partial\Gamma}$$

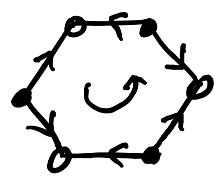
<u>Proof</u> h/w

i.e. Boundary value of each h.fncn in $\mathcal{H}_\Gamma$ depend only on Γ and is the same.

Thm. The probability of a dimer configuration can be written as

$$\mathrm{Prob}(D) = \frac{1}{Z} \prod_f q_f^{\,h_D(f)} \qquad (*)$$

$$q_f = \prod_{e \in \partial f} w_e^{\,\varepsilon(e)}$$



Proof h/w

Remark.

- $\mathrm{Prob}(D)$ is invariant $w(e) \mapsto s(e_+)\, w(e)\, s(e_-)$
- q_f are invariant ("essential" parameters)

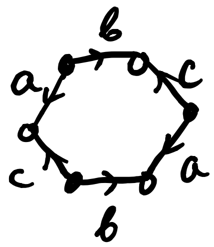
Dimer probability measure $(*)$ + this bijection gives

$$\mathrm{Prob}(h) = \frac{1}{Z} \prod_{f \in \Gamma \subset \mathbb{R}^2} q_f^{h(f)}, \quad h \in \mathcal{H}_\Gamma$$

$$Z = \sum_{h \in \mathcal{H}_\Gamma} \prod_f q_f^{h(f)}$$

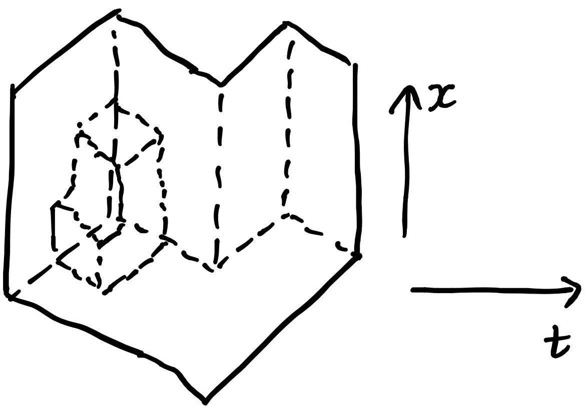
$h|_{\partial\Gamma}$ = determined by Γ only

Particular cases:

(1) 

$$q = a^{-1} b c^{-1} a b^{-1} c = 1$$

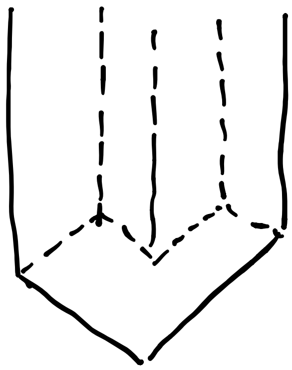
the uniform distribution

(2) 

$$q_f = q_x$$

$$\mathrm{Prob}(h) \propto \prod_t q_t^{\pi(t)}$$

If $0 < q_t < 1$ one can allow unbounded piles



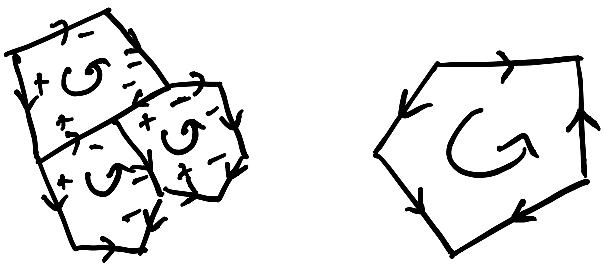
$$\mathrm{Prob}(\pi) \propto \prod_t q_t^{\pi(t)}$$

Kasteleyn solution

Computation of the partition function and of correlation functions in terms of Pfaffians.

$(\Gamma \subset \mathbb{R}^2)$

① Kasteleyn orientation of a plane graph.

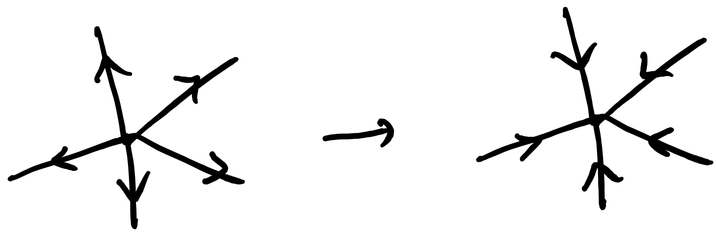


$$\prod_{e \in \partial f} \varepsilon(e,f) = -1$$

Proposition Kasteleyn orientations exist

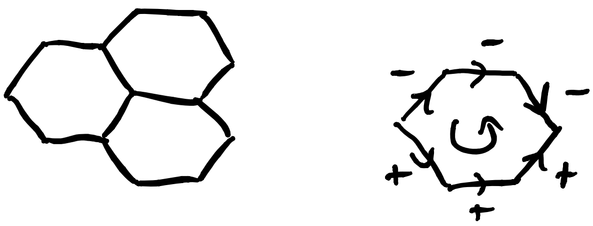
Proof. h/w

Def. Two K-orientations K, K' are equivalent if K' can be obtained from K by a sequence of orientation reversing maps:

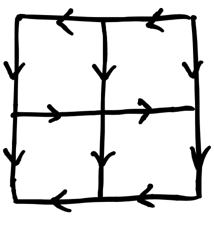


Thm. All K-orientations of a plane graph are equivalent.

Ex. 1)



2)



② Kasteleyn matrix:

Let $w(e) > 0$ be weights of edges and K be a Kasteleyn orientation

$$A^K_{ij} = \begin{cases} w(ij) , & i \bullet\!\!\rightarrow\!\!\bullet j \\ -w(ij) , & i \bullet\!\!\leftarrow\!\!\bullet j \\ 0 , & \text{not connected} \end{cases}$$

<u>Theorem</u> (Kasteleyn)

$$\sum_{D \subset \Gamma} \prod_{e \in D} w(e) = | \mathrm{Pf}(A^K) |$$

where

$$\mathrm{Pf}(A) = \frac{1}{2^{n/2} (n/2)!} \sum_{\sigma \in S_n} (-1)^{\sigma} \prod_{i=1}^{n-1} A_{\sigma_i \sigma_{i+1}}$$

A : $n \times n$, skew symmetric, n is even

<u>The idea of the proof</u>

- Dimers = perfect matchings on edges connected by edges

$$Z = \sum_{m=(i_1 i_2)\cdots(i_{n-1} i_n)} w(i_1 i_2)\dots\; w(i_{n-1}\, i_n)$$

in a perfect matching

$$\dots(i_{k-1}\, i_k)\dots \equiv \dots (i_k\, i_{k-1})\dots$$

$$\cdot\cdot(i_{\ell-1} i_\ell)\cdots(i_{k-1}, i_k)\cdots \equiv \cdots (i_{k-1} i_k)\cdots (i_{\ell-1} i_\ell)\cdots$$

$S_2^{\frac{n}{2}}$ $\qquad$ $S_{\frac{n}{2}}$

$$\Rightarrow \text{perfect matchings} \simeq S_n / S_2^{n/2} \times S_{\frac{n}{2}}$$

$$Pf(A^K) = \sum_{\substack{m \in S_n/S_2^{\frac{n}{2}} \times S_{\frac{n}{2}} \\ [\sigma]}} \underbrace{(-1)^\sigma A^K_{\sigma_1\sigma_2}\cdots A^K_{\sigma_{n-1}\sigma_n}}_{\text{depends only on } m,\text{ not on } \sigma\in m}$$

$$= \sum_{m=[\sigma]} \overbrace{(-1)^\sigma \varepsilon^K_{\sigma_1\sigma_2}\cdots\varepsilon^K_{\sigma_{n-1}\sigma_n}} \prod_{i=1}^{n-1} w(m_i, m_{i+1})$$

- **Lemma** $(-1)^{\sigma} \varepsilon^{K}_{\sigma_1\sigma_2} \cdots \varepsilon^{K}_{\sigma_{n-1}\sigma_n}$ does not depend on $m = [\sigma]$

- **Corollary**

$$\mathrm{Pf}(A^K) = \underbrace{(\text{sign})}_{= (-1)^{\sigma} \varepsilon^{K}_{\sigma_1\sigma_2} \cdots \varepsilon^{K}_{\sigma_{n-1}\sigma_n}} \cdot Z$$

for all σ

$\blacksquare$

Corollary. From the Pfaffian formula for the partition function

$$\langle \sigma_{i_1 j_1} \cdots \sigma_{i_k j_k} \rangle =$$

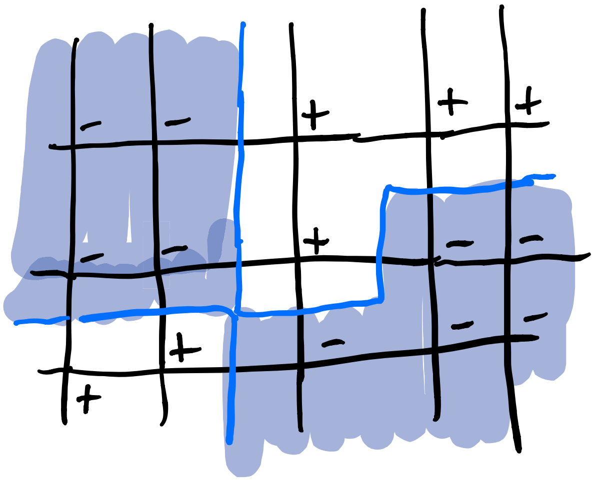
$$= \mathrm{Pf}\left((A^K)^{-1}_{ab} \right)_{a,b \in \{i_1, \ldots, i_k, j_1, \ldots, j_k\}}$$

h/w. Prove this

Thus :

- Partition fncn = $\mathrm{Pf}(A^K)$
- Correlation fncns = $(A^K)^{-1}$

<u>Local mapping of Ising model to dimers</u>



Clusters of + and -

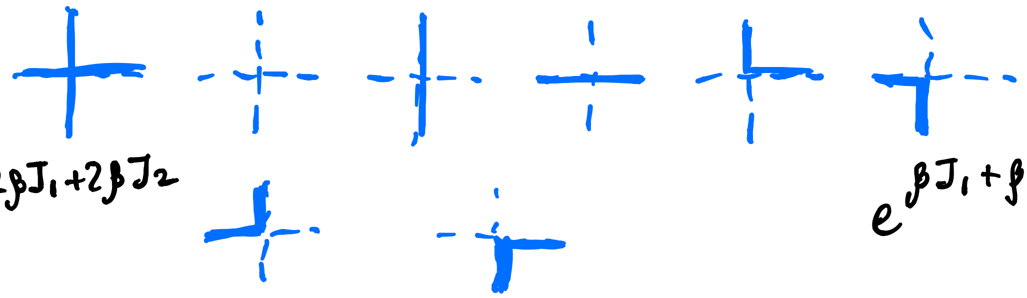
Let $H = 0$

$$E(\sigma) = J_1 \sum_{\substack{\langle ij \rangle \\ \text{vertical nbrs}}} \sigma_i \sigma_j + J_2 \sum_{\substack{\langle ij \rangle \\ \text{horizontal nbrs}}} \sigma_i \sigma_j$$

$$e^{-\beta E(\sigma)} = e^{-\beta J_1 |E_v| - \beta J_2 |E_h|} \prod_{e_v \in C} e^{2\beta J_1} \prod_{e_h \in C} e^{2\beta J_2}$$

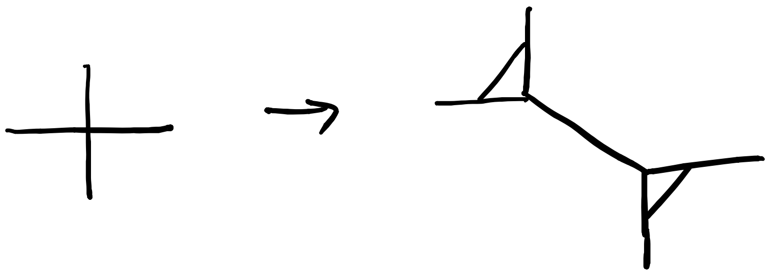
Locally on the dual lattice $\Gamma^\vee$:

Cluster path configurations

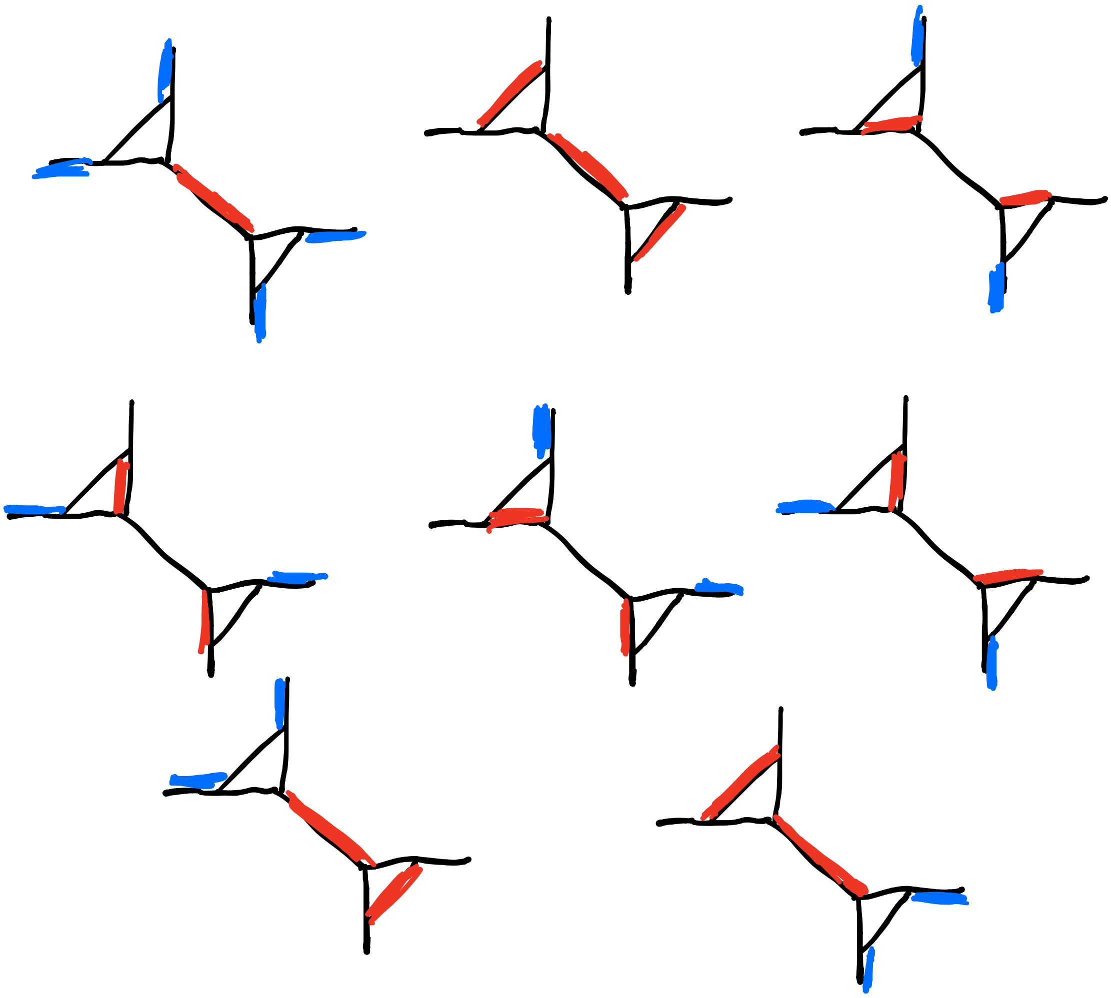


$e^{2\beta J_1 + 2\beta J_2}$ $\qquad$ $e^{\beta J_1 + \beta J_2}$

Stretch the lattice:



Then Ising cluster path config. become:



h/w find dimer weights which will produce Ising weights

Corollary

$$Z_{\mathrm{Ising}_{\Gamma}} = Z_{\mathrm{Dimer}_{\tilde{\Gamma}}} = \left|\mathrm{Pf}\left(A^{K}_{\tilde{\Gamma}}\right)\right|$$

Correlation functions $= \left(A^{K}_{\tilde{\Gamma}}\right)^{-1}$

This is equivalent to the Onsager-Kauffmann solution.