PROBLEMS FOR BEZRUKAVNIKOV AND BRAVERMAN LECTURES

1. Let X be a smooth algebraic curve over $\mathbb{C}$ with a point $x \in X$. Choose a local (formal) coordinate t near x.

a) Show that the affine Grassmannian Gr_G classifies pairs $(\mathcal{P}, \phi)$ where $\mathcal{P}$ is a principal G-bundle on X and ϕ is a trivialization of $\mathcal{P}$ on $X \backslash \{x\}$.

b) Let $X = \mathbb{P}^1$. Let Gr_G^0 denote the subset of Gr_G consisting of those pairs $(\mathcal{P}, \phi)$ where $\mathcal{P}$ is trivial. Show that Gr_G^0 is an open subset of Gr_G. Show that it is isomorphic to $G[t^{-1}]/G$ (which naturally embeds into $G((t))/G[[t]])$).

c) Invent a similar interpretation of the affine flag variety of G.

2. In this problem for simplicity we take $G = SL(n)$. Recall that the determinant line bundle $\mathcal{L}$ on Gr_G is defined as follows: given $g \in SL(n, \mathcal{K})$ set $L_0 = \mathcal{O}^n, L = g(L_0)$. The the fiber of $\mathcal{L}$ at $g \mod G(\mathcal{O})$ is equal to $\det(L/L \cap L_0)^* \otimes \det(L_0/L \cap L_0)$ (here $*$ stands for dual line; this definition is actually dual to the one discussed in the lecture).

a) Let us use the notation of Problem 1. Express the fiber of $\mathcal{L}$ at $(\mathcal{P}, \phi)$ in terms of determinant of cohomology of the bundle $\mathcal{P}$ (which in the case $G = SL(n)$ can be thought of as rank n vector bundle with trivial determinant).

b) Take $X = \mathbb{P}^1$. Use a) in order to construct a trivialization of $\mathcal{L}$ on Gr_G on Gr_G^0.

c)* Show that the above trivialization extends to a section of $\mathcal{L}$ on all of Gr_G whose zero locus is precisely the complement of Gr_G^0.

3. Let $G = SL(2)$. Compute the stabilizer of a point $t^\lambda \in \mathrm{Gr}_{SL(2)}$ (say, for a dominant coweight λ) in $SL(2, \mathcal{O})$. Show that unless $\lambda = 0$ this stabilizer acts non-trivially in the fiber of the determinant bundle $\mathcal{L}$. Deduce from this that if q is not a root of 1, then the category of q-monodromic $SL(2, \mathcal{O})$-equivariant sheaves on $\mathcal{L}^0$ (total space of $\mathcal{L}$ with zero-section removed) is equivalent to the category of vector spaces.

4. Let $N \subset G$ be a maximal unipotent subgroup of G normalized by a maximal torus T.

a) Show that the assignment $\lambda \mapsto N(\mathcal{K}) \cdot t^\lambda$ defines a one-to-one correspondence between coweights of T and $N(\mathcal{K})$-orbits on Gr_G. Set $S^\lambda = N(\mathcal{K}) \cdot t^\lambda$.

b) Let χ_0 be a non-degenerate character of N. Let $\chi : N(\mathcal{K}) \to \mathbb{G}_a$ be Show that S^λ carries a non-zero $N(\mathcal{K}), \chi)$-equivariant D-module iff λ is dominant (hint: prove first the following fact. Let H be an algebraic group acting transitively on a variety X and let $\chi : H \to \mathbb{G}_a$ be any homomorphism. Then X carries a non-zero (H, χ)-equivariant D-module iff χ is trivial on the stabilizer of a point $x \in X$ in H). Explain (informally) why this means that "size-wise" the category $Whit(\mathrm{Gr}_G)$ looks like $\mathrm{Rep}(G^\vee)$.

4. Let G be a simple algebraic group. Consider the functor $I_c : \mathrm{Rep}(G) \to \mathrm{KL}_c(G)$ which sends a G-module M to $(U(\hat{\mathfrak{g}}) \underset{U(\mathfrak{g})}{\otimes} M)/(\mathbf{1} - (c - h^\vee))$.

a) Show that this functor indeed lands in $\mathrm{KL}_c(G)$.

b) Show that this functor is an equivalence of categories $Rep(G) \simeq \mathrm{KL}_c(G)$ if c is not rational. In particular, $KL_c(G)$ is a semi-simple category in this case.

c) Produce an explicit example which shows that b) is false for rational c.