INTRODUCTION TO POISSON LIE GROUPS, LIE BIALGEBRAS AND THEIR QUANTIZATION

1. Let G be a Poisson-Lie group, and let $G^{(0)} = \{x \in G : \Pi(x) = 0\}$ be the subset of all points where the Poisson structure vanishes. Show that $G^{(0)}$ is a closed Lie subgroup of G.

2. Let G be a Poisson-Lie group with Lie bialgebra $(\mathfrak{g}, [,], \delta)$. Let $H \subset G$ be a closed, connected Lie subgroup and let $Lie(H) = \mathfrak{h} \subset \mathfrak{g}$. Show that G/H inherits a unique Poisson structure such that $\pi : G \to G/H$ is a Poisson map if and only if $\delta(\mathfrak{h}) \subset \mathfrak{h} \otimes \mathfrak{g} + \mathfrak{g} \otimes \mathfrak{h}$.

3. If G_1 and G_2 are simply connected Poisson-Lie groups and $\mathfrak{g}_1 = \widetilde{F}(G_1)$, $\mathfrak{g}_2 = \widetilde{F}(G_2)$, then there is a one-to-one correspondence between the Poisson-Lie group morphisms $G_1 \to G_2$ and the Lie bialgebra morphisms $\mathfrak{g}_1 \to \mathfrak{g}_2$.

4. Show that any coboundary Lie bialgebra structure on $\mathfrak{a} = \mathbb{C}[[x]]\frac{d}{dx}$ of formal vector fields on the line can be obtained as an image of the $\tilde{b}_2$ Lie bialgebra structure of T_2 under a homomorphism $T_2 \to \mathfrak{a}$, and classify all such structures up to isomorphism.

5. Let V be a finite-dimensional vector space. We say that a Poisson bracket on V is of degree n if the Poisson bivector is a polynomial on V of degree n. Poisson brackets of degrees $0, 1, 2$ are called constant, linear and quadratic, respectively.

Let A be an associative, finite dimensional algebra with unit over $\mathbb{R}$, and let $\{,\}$ be a Poisson bracket on A, such that the multiplication map $m : A \times A \to A$ is a Poisson map. Show that the Poisson bracket is quadratic.

6. Recall the construction of $\widetilde{\mathfrak{g}}$ for a simple Lie algebra $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ with Killing form $(\cdot,\cdot) = (\cdot,\cdot)_{\mathfrak{g}}$. As a vector subspace, $\widetilde{\mathfrak{g}} = \mathfrak{n}_+ \oplus \mathfrak{h}^{(1)} \oplus \mathfrak{h}^{(2)} \oplus \mathfrak{n}_-$, where $\mathfrak{h}^{(1)} \cong \mathfrak{h}^{(2)} \cong \mathfrak{h}$. Its Lie bracket is determined by the relations

$$[\mathfrak{h}^{(1)}, \mathfrak{h}^{(2)}] = 0, \qquad [h^{(i)}, e_\alpha] = \alpha(h) e_\alpha \qquad [h^{(i)}, f_\alpha] = -\alpha(h) f_\alpha, \qquad [e_\alpha, f_\alpha] = \frac{1}{2}(h_\alpha^{(1)} + h_\alpha^{(2)}).$$

The Lie algebra $\widetilde{\mathfrak{g}}$ can be given the nondegenerate invariant bilinear form $(\cdot,\cdot)_{\widetilde{\mathfrak{g}}}$ defined as follows

$$(x + h^{(1)} + h^{(2)}, x' + h'^{(1)} + h'^{(2)})_{\widetilde{\mathfrak{g}}} = 2((h^{(1)}, h'^{(2)}) + (h^{(2)}, h'^{(1)})) + (x, x').$$

Show that $\widetilde{\mathfrak{g}}$ is naturally isomorphic to the Lie algebra $\mathfrak{g} \oplus \mathfrak{h}$, and that the invariant form for $\widetilde{\mathfrak{g}}$ given above is mapped under this isomorphism to the form $(,)_{\mathfrak{g}} - (,)_{\mathfrak{h}}$.

7. Let A be a finite dimensional commutative algebra which is Frobenius, this is, there exists a trace map $tr : A \to k$ such that the bilinear form $(a, b)_A = tr(ab)$ is nondegenerate. Let $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ be a Manin triple with bilinear form $(,)_{\mathfrak{g}}$. Let $\mathfrak{g}_A = \mathfrak{g} \otimes_k A$ and $\mathfrak{g}_{\pm,A} = \mathfrak{g}_\pm \otimes_k A$. Then the bilinear form $(,)_{\mathfrak{g}} \otimes (,)_A$ on $\mathfrak{g}_A$ is invariant and nondegenerate, and $(\mathfrak{g}_A, \mathfrak{g}_{+,A}, \mathfrak{g}_{-,A})$ is a Manin triple.

8. Let $\mathfrak{g}$ be a Lie bialgebra and $r \in \mathfrak{g} \oplus \mathfrak{g}$ a solution to the CYBE. Let

$$\begin{aligned} \mathfrak{g}_r^+ &= Span\{(1 \otimes f) r | f \in \mathfrak{g}^*\}, \\ \mathfrak{g}_r^- &= Span\{(f \otimes 1) r | f \in \mathfrak{g}^*\}. \end{aligned}$$

Then $\mathfrak{g}_r^+$ and $\mathfrak{g}_r^-$ are finite dimensional Lie subcoalgebras of $\mathfrak{g}$.