BPS/CFT correspondence and qq-characters: exercises

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Problems for days 1-3

• P1: Using the argument with decoupling of Q-exact terms prove the main theorem of algebra: *degree n polynomial with complex coefficients has n complex roots*. What goes wrong with the same argument in the real case?

• P2: Using the argument with decoupling of Q-exact terms relate the sum of indices of zeroes of a vector field V on a compact manifold M to the integral of the Pfaffian of a curvature form R for some Riemannian metric on M.

• P3: Using Duistermaat-Heckmann formula compute the $U(1) \times SU(n)$ equivariant symplectic volume of $T^*Gr(k,n)$. By taking the appropriate limits find the symplectic volume of the Grassmanian $Gr(k,n)$.

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Problems for days 1-3

- P4: Assume ADHM equations on (B_1, B_2, I, J) at zero level of the moment maps. Verify that the connection given by the formula

$$\mathbf{A} = \mathbf{\Psi}^{\dagger} d\mathbf{\Psi}$$

where $\mathbf{\Psi} : N \to N \oplus K \oplus K$ is the normalized, $\mathbf{\Psi}^{\dagger}\mathbf{\Psi} = 1_N$, solution to $\mathcal{D}^{\dagger}\mathbf{\Psi} = 0$, where $\mathcal{D}^{\dagger} : N \oplus K \oplus K \to K \oplus K$ is given by

$$\mathcal{D}^{\dagger} = \begin{pmatrix} B_1 - z_1 \cdot 1_K & B_2 - z_2 \cdot 1_K & I \\ \bar{z}_2 \cdot 1_K - B_2^{\dagger} & B_1^{\dagger} - \bar{z}_1 \cdot 1_K & J^{\dagger} \end{pmatrix}$$

is an instanton, i.e. obeys

$$F^{+} = 0\,, \ \ F = d\mathbf{A} + \mathbf{A} \wedge \mathbf{A}$$

Hint: the space of self-dual two forms on $\mathbb{R}^4 \approx \mathbb{C}^2$ has a basis spanned by:

$$\varpi_{\mathbb{R}} = \frac{\mathrm{i}}{2}\left(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2\right)\,,\ \varpi_{\mathbb{C}} = dz_1 \wedge dz_2\,,\ \bar{\varpi}_{\mathbb{C}} = d\bar{z}_1 \wedge d\bar{z}_2$$

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Problems for days 1-3

• P5: Now turn on $\vec{\zeta} = (\zeta_{\mathbb{R}} = r > 0, 0, 0)$. Modify the commutation relations between the coordinates $(z_1, z_2, \bar{z}_1, \bar{z}_2)$ so that $P3$ still holds.

• P6: Compute **A** for $K = \mathbb{C}^1$, $N = \mathbb{C}^1$. Represent the commutation relations between $(z_1, z_2, \bar{z}_1, \bar{z}_2)$ using: a) Two Heisenberg commuting algebras:

$$[a_i, a_j^\dagger] = \delta_{i,j}\ ,\ \ i, j = 1, 2$$

b) One commutative $[z_1, \bar{z}_1] = 0$ and one Heisenberg algebra $[a, a^\dagger] = 1$. Where hides the topology of the instanton solution?

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Problems for days 3-4

• P7: Compute the **T**-character of the tautological ADHM complex, at the fixed point $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(n)}) \in \mathcal{M}(k, n)$:

$$S = N - P_{12}K = S^+ - S^-$$

using $P_{12} = (1 - q_1)(1 - q_2)$, and

$$N = \sum_{\alpha=1}^{n} e^{a_\alpha}\ , \ \ K = \sum_{\alpha=1}^{n} e^{a_\alpha} \sum_{(i,j)\in\lambda^{(\alpha)}} q_1^{i-1} q_2^{j-1}$$

Identify the pure characters S^+, S^- with the characters of H^1 and H^0 cohomology groups. Identify (with the help of Young diagrams of $\lambda^{(\alpha)}$) the individual **T**-eigenspaces.

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- P8: Compute $Y(x)|_\lambda$ for $\lambda = (\Box, 0, 0, \ldots, 0)$.

- P9: Compute the $\mathbf{T}$-character of the tangent space $T_\lambda \mathcal{M}(k,n)$ at the fixed point $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(n)}) \in \mathcal{M}(k,n)$:

$$T = NK^* + q_1 q_2 N^* K - P_{12} K K^*$$

Again, identify (with the help of Young diagrams of $\lambda^{(\alpha)}$) the individual $\mathbf{T}$-eigenspaces.

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