Problems from A.O.'s lectures

1. Let Y be the variety parametrizing all partial flags in $\mathbb{C}^n$ of the form

$$V_1 \subset V_2 \subset \cdots \subset V_k \subset V_{k+1} = \mathbb{C}^n$$

Realize T^*Y as Nakajima quiver variety for the linear quiver of length k.

2. What happens in this example as we vary the stability conditions ?

3. Prove that any Nakajima quiver variety is smooth.

4. Prove that any Nakajima quiver variety is symplectic.

5. Compute the dimension of a Nakajima quiver variety (whenever nonempty)

6. Compute the K-theory class of the tangent bundle to a Nakajima quiver variety X in terms of the tautological bundles on X. What does your formula say for $T^*\mathbb{P}^n$ and for $T^* \operatorname{Gr}(k,n)$?

7. Prove that the Hilbert scheme of n points in $\mathbb{C}^2$ is a Nakajima quiver variety.

8. Let A be a torus acting on a Nakajima quiver variety X via an action on the framing spaces W_i. Prove that the fixed locus X^{A} is a union of products of Nakajima varieties associated to the same quiver.

9. Compute the normal bundle to the fixed locus in the previous problem as an element of $K_A(X^A)$.

10. Let $\operatorname{Hom}(V_i, V_j)$ be one of the arrows in the quiver and let $a \in \mathbb{C}^\times$ rescale it with weight a, while rescaling the opposite arrow $\operatorname{Hom}(V_j, V_i)$ with the opposite weight a^{-1}. Prove that this gives a well-defined action that preserves the symplectic form. Compute X^a. Describe the action of a on the restriction of tautological bundles to X^a.

11. What does the previous problem say for the Hilbert scheme of points ? Let T be the maximal torus of $GL(2)$. Compute the character of T-action on the tangent spaces to its fixed points on $\operatorname{Hilb}(\mathbb{C}^2, n)$.

12. Consider the quiver with one vertex and one loop, and let the framing space $W = \mathbb{C}^r$ be r dimensional. Compute the fixed loci for the maximal torus of $GL(W)$ and the normal bundle to them.

13. Same for the maximal torus of $GL(2) \times GL(W)$.

14. What does the Grauert-Riemenschneider theorem say about the map

$$\mathrm{Hilb}(\mathbb{C}^2, n) \to S^n\mathbb{C}^2 \quad ?$$

Compute the character of global functions on the Hilbert scheme of points using this map.

15. What does equivariant localization say about the character of global functions on $\mathrm{Hilb}(\mathbb{C}^2, n)$? Check your answer against the answer computed in the previous problem using any computer algebra system.

16. Denote $\mathscr{O} = \mathbb{C}[x_1, \ldots, x_n]$ and let

$$\mathfrak{m} \subset \mathscr{O}$$

be the ideal generated by $(x_1, \ldots, x_n)$. Construct a $GL(n)$-equivariant free resolution of $\mathfrak{m}$ and $\mathfrak{m}^2$.

17. Let X be smooth and let

$$\iota : Y \to X$$

be an inclusion of a smooth subvariety. Compute $\iota^*\iota_*\mathscr{O}_Y$ in $K(Y)$.

18. Let X be union $x_1x_2 = 0$ of two lines in a plane. And let

$$\mathscr{O}_0 = \mathscr{O}_X/(x_1, x_2)$$

be the structure sheaf of the origin. Compute the minimal free resolution of $\mathscr{O}_0$.

19. Consider the nonequivariant K-theory of X and let

$$K(X) \ni \mathscr{F} \mapsto \mathscr{E} \otimes \mathscr{F} \in K(X)$$

be the operator of tensor multiplication by a vector bundle of rank r. Prove that r is the only eigenvalue of this operator.

20. What are the eigenvalues of this operator in equivariant K-theory ?

21. Compute the cohomology of $\mathscr{O}(k)$ on $\mathbb{P}^n$.

22. Can you see directly from localization formula that the $\chi_{\mathbb{P}^n}(\mathscr{O}(k))$ vanishes for $k = -1, \dots, -n$?

23. Let $X = G/B$ be the flag variety of a reductive group G. For every character

$$\lambda : B \to \mathbb{C}^\times$$

there is an associates line bundle $\mathscr{L}_\lambda$ on X. Compute the Euler characteristic $\chi(\mathscr{L}_\lambda)$ using localization with respect to a maximal torus of B.

24. Let Ω^k denote $\Lambda^k T^*$, that is, the kth exterior power of the tangent bundle. Show that the generating function

$$\sum_{n,k} z^n \, (-m)^k \, \chi(\mathrm{Hilb}(\mathbb{C}^2, n), \Omega^k)$$

factors in a nice infinite product, which may be interpreted as the symmetric algebra of a virtual vector space with character

$$\frac{z}{1 - mz} \, \chi(\mathrm{Hilb}(\mathbb{C}^2, 1), \Omega^0 - \Omega^1 + \Omega^2) \, .$$