# Problems from A.O.'s lectures, part 1

**1.** Let $Y$ be the variety parametrizing all partial flags in $\mathbb{C}^n$ of the form

$$V_1 \subset V_2 \subset \cdots \subset V_k \subset V_{k+1} = \mathbb{C}^n$$

Realize $T^*Y$ as Nakajima quiver variety for the linear quiver of length $k$.

**2.** What happens in this example as we vary the stability conditions ?

**3.** Prove that any Nakajima quiver variety is smooth.

**4.** Prove that any Nakajima quiver variety is symplectic.

**5.** Compute the dimension of a Nakajima quiver variety (whenever nonempty)

**6.** Compute the K-theory class of the tangent bundle to a Nakajima quiver variety $X$ in terms of the tautological bundles on $X$. What does your formula say for $T^*\mathbb{P}^n$ and for $T^*\operatorname{Gr}(k,n)$ ?

**7.** Prove that the Hilbert scheme of $n$ points in $\mathbb{C}^2$ is a Nakajima quiver variety.

**8.** Let $\mathsf{A}$ be a torus acting on a Nakajima quiver variety $X$ via an action on the framing spaces $W_i$. Prove that the fixed locus $X^{\mathsf{A}}$ is a union of products of Nakajima varieties associated to the same quiver.

**9.** Compute the normal bundle to the fixed locus in the previous problem as an element of $K_A(X^A)$.

**10.** Let $\operatorname{Hom}(V_i, V_j)$ be one of the arrows in the quiver and let $a \in \mathbb{C}^\times$ rescale it with weight $a$, while rescaling the opposite arrow $\operatorname{Hom}(V_j, V_i)$ with the opposite weight $a^{-1}$. Prove that this gives a well-defined action that preserves the symplectic form. Compute $X^a$. Describe the action of $a$ on the restriction of tautological bundles to $X^a$.

**11.** What does the previous problem say for the Hilbert scheme of points ? Let $\mathsf{T}$ be the maximal torus of $GL(2)$. Compute the character of $\mathsf{T}$-action on the tangent spaces to its fixed points on $\operatorname{Hilb}(\mathbb{C}^2, n)$.

**12.** Consider the quiver with one vertex and one loop, and let the framing space $W = \mathbb{C}^r$ be $r$ dimensional. Compute the fixed loci for the maximal torus of $GL(W)$ and the normal bundle to them.

**13.** Same for the maximal torus of $GL(2) \times GL(W)$.

**14.** What does the Grauert-Riemenschneider theorem say about the map

$$\mathrm{Hilb}(\mathbb{C}^2, n) \to S^n \mathbb{C}^2 \quad ?$$

Compute the character of global functions on the Hilbert scheme of points using this map.

**15.** What does equivariant localization say about the character of global functions on $\mathrm{Hilb}(\mathbb{C}^2, n)$ ? Check your answer against the answer computed in the previous problem using any computer algebra system.