Chern-Simons gauge theory as a string theory

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1 Introduction

String theory can be regarded, at the algebraic level, as a two-dimensional conformal field theory coupled to two-dimensional gravity [1, 2]. When the conformal field theory is also a topological field theory (i.e. a theory whose correlation functions do not depend on the metric on the Riemann surface), the resulting string theory turns out to be very simple and in many cases can be completely solved. A string theory that is constructed in this way is called a topological string theory.

The starting point to obtain a topological string theory is therefore a conformal field theory with topological invariance. Such theories are called topological conformal field theories and can be constructed out of $\mathcal{N} = (2, 2)$ superconformal field theories in two dimensions by a procedure called twisting [3, 4]. In this note, we will consider a class of topological string theories in which the topological field theory is taken to be a topological sigma model with target space a Calabi–Yau manifold. In two dimensions there are two possible twists, leading respectively to the A-type and the B-type topological sigma models, which we study in detail in this chapter.

One important question that we will address is the following: Is it possible to make more precise this analogy between $U(N)$ gauge theories and open string theories? In other words, given a $U(N)$ gauge theory, is it possible to find an open string theory in such a way that the quantities $F_{g,h}$ that appear in expansion of free energy can be interpreted as open string amplitudes, for some open string and some target manifold? In some cases the answer is yes, and involves open topological strings whose target is a Calabi–Yau manifold and whose boundary conditions are specified by topological D-branes. The main tool to provide this identification is string field theory, i.e. a field theory defined on the target of the string that describes its spacetime dynamics. The open string field theory we will need was introduced by Witten [5]. Although it was originally constructed for the open bosonic string theory, it can also be applied to topological string theory, and it turns out that on some particular Calabi–Yau backgrounds the full string field theory of the topological string reduces to a simple $U(N)$ gauge theory, where g_s plays the role of the gauge coupling constant and N is the rank of the gauge group. In particular, the string field reduces in this case to a finite number of gauge fields. For topological string theories exist two famous examples:

1) The type-A model on a Calabi-Yau of the form $X = T^*M$, where M is a three-manifold, and there are N topological D-branes wrapping M. In this case, the gauge theory is Chern-Simons theory on M [6].

2) The B model on a Calabi-Yau manifold X that is obtained as the small resolution of the singularity $y^2 = (W'(x))^2$. If $W(x)$ has degree n, the small resolution produces n two-spheres, and one can wrap N_i topological D-branes around each two-sphere, with $i = 1, \ldots, n$. In this case, the gauge theory is a multicut matrix model with potential $W(x)$ [7].

In this note, We will focus on first example and describe how Chern-Simons gauge theory in three dimensions can be viewed as a string theory [6]. The perturbation theory of this string theory will coincide with Chern-Simons perturbation theory [8]. Chern-Simons theory enters in this particular string theory in much the same way that ordinary space-time physics (with general relativity as the long wavelength limit) arises in conventional string theory [1, 2].

2 String field theory and gauge theories

In this section, following the program sketched in the introduction, we show that Chern–Simons gauge theories can be realized as open string theories.

2.1 Open string field theory

As we explained in the previous chapter, our stategy to show that Chern-Simons theory are open string theories is to show that these gauge theories describe the spacetime dynamics of topological open strings on certain backgrounds, and to do this we will use string field theory. We briefly summarize here some basic ingredients of the cubic string field theory introduced by Witten [5, 9] to describe the spacetime dynamics of open bosonic strings, since we will use the same model to describe topological strings. In bosonic open string field theory, we consider the worldsheet of the string to be an infinite strip parameterized by a spatial co-ordinate $0 \leq \sigma \leq \pi$ and a time co-ordinate $-\infty < \tau < \infty$, and we pick the flat metric $ds^2 = d\sigma^2 + d\tau^2$. We then consider maps $x : I \to X$ with $I = [0, \pi]$ and X the target of the string. The string field is a functional of open string configurations $\Psi[x(\sigma)]$, with *ghost number* one (although we will not indicate it explicitly, this string functional depends on the ghost fields as well). Witten defines two operations on the space of string functionals. The first one is the *integration*, which is defined formally by folding the string around its midpoint and gluing the two halves:

$$\int \Psi = \int Dx(\sigma) \prod_{0 \leq \sigma \leq \pi} \delta[x(\sigma) - x(\pi - \sigma)] \Psi[x(\sigma)] \tag{1}$$

The integration has ghost number -3, which is the ghost number of the vacuum. This corresponds to the usual fact that in open string theory on the disc one has to soak up three zero modes. One also defines an associative, non-commutative *star product* $\star$ of string functionals through the following equation:

$$\int \Psi_1 \star \cdots \star \Psi_N = \int \prod_{i=1}^{N} Dx_i(\sigma) \prod_{i=1}^{N} \prod_{0 \leq \sigma \leq \pi} \delta[x_i(\sigma) - x_{i+1}(\pi - \sigma)] \Psi_i[x_i(\sigma)] \tag{2}$$

where $x_{N+1} \equiv x_1$. The star product simply glues the strings together by folding them around their midpoints, and gluing the first half of one with the second half of the following one, and it doesn't change the ghost number. In terms of these geometric operations, the string field action is given by

$$S = \frac{1}{g_s} \int \left(\Psi \star Q_{BRST} \Psi + \frac{2}{3} \Psi \star \Psi \star \Psi \right) \tag{3}$$

Notice that the integrand has ghost number 3, while the integration has ghost number -3, so that the action has zero ghost number. If we add Chan-Paton factors, the string field is promoted to a U(N) matrix of string fields, and the integration in includes a trace Tr. The action has all the information about the spacetime dynamics of open bosonic strings, with or without D-branes. In particular, one can derive the Born–Infeld action describing the dynamics of D-branes from the above action [10].

We will not need all the technology of string field theory in order to understand open topological strings. The only piece of relevant information is the following: the string functional is a function of the zero mode of the string (which corresponds to the position of the string midpoint), and of the higher oscillators. If we decouple all the oscillators, the string functional becomes an ordinary function of spacetime, the $\star$ product becomes the usual product of functions, and the integral is the usual integration of functions. The decoupling of the oscillators is, in fact, the pointlike limit of string theory. As we will see, this is the relevant limit for topological open strings.

2.2 Chern–Simons theory as an open string theory

Let M be an arbitrary (real) three-dimensional manifold, and consider the six-dimensional space given by the cotangent bundle T^*M of M. This space is a symplectic manifold. If we pick local co-ordinates q_a on M, $a = 1, 2, 3$, and local co-ordinates for the fibre p_a, the symplectic form can be written as

$$J = \sum_{a=1}^{3} dp_a \wedge dq_a \tag{4}$$

One can find a complex structure on T^*M such that J is a Kahler form, so T^*M can be regarded as a Kahler manifold. Since the curvature of the cotangent bundle exactly cancels the curvature of M, it is Ricci-flat, therefore it is a Calabi-Yau manifold. An important example is the cotangent bundle of the three-sphere, T^*S^3, which can be described holomorphically as the deformed conifold.

It is obvious that M is a Lagrangian submanifold in T^*M, since J vanishes along $p_a = const.$ Since we have a Calabi–Yau manifold together with a Lagrangian submanifold in it, we can consider a system of N topological D-branes wrapping M, thus providing Dirichlet boundary conditions for topological open strings on T^*M. Our goal now is to obtain a spacetime action describing the dynamics of these topological D-branes, and as we will see this action is simply Chern-Simons theory on M. This will prove to be the sought-for realization of Chern-Simons theory in terms of open strings.

In order to find the spacetime action, we exploit again the *analogy between open topological strings and the open bosonic string* that we used to define the coupling of topological sigma models to gravity (i.e. that both have a nilpotent BRST operator and an energy-momentum tensor that is Q_{BRST} - exact). Since both theories have a similar structure, the spacetime dynamics of topological D-branes in T^*M is also governed by 3, where Q_{BRST} is given in this case by the topological charge, and where the star product and the integration operation are as in the bosonic string. The construction of the cubic string field theory also requires the existence of a ghost-number symmetry, which is also present in the topological sigma model. Sometimes it is convenient to redefine the ghost number by shifting it by -d/2 units with respect to the assignment presented in Chapter 3 (here, d is the dimension of the target). When d = 3 this corresponds to the normalization used by Witten in which the ghost vacuum of the bc system is assigned the ghost number -1/2.

In order to provide the string field theory description of open topological strings on T^*M, we have to determine the precise content of the string field, the $\star$ algebra and the integration of string functionals for this particular model. As in the conventional string field theory of the bosonic string, we have to consider the Hamiltonian description of topological open strings. We then take Σ to be an infinite strip and consider maps $x : I \to T^*M$, with $I = [0, \pi]$, such that ∂I is mapped to M. The action is taken to be tS_A:

$$L_A = -2t \left(g_{i\bar{j}} \partial_z \phi^i \partial_{\bar{z}} \bar{\phi}^j + g_{i\bar{j}} \partial_{\bar{z}} \phi^i \partial_z \bar{\phi}^j + i g_{i\bar{j}} \psi^i_z \Delta_{\bar{z}} \chi^{\bar{j}} + i g_{i\bar{j}} \psi^{\bar{j}}_{\bar{z}} \Delta_z \chi^i + \frac{1}{2} R_{i\bar{j}k\bar{l}} \psi^i_z \psi^{\bar{j}}_{\bar{z}} \chi^k \chi^{\bar{l}} \right) \tag{5}$$

$$L' = -it\{Q_A, V\} \qquad V = g_{i\bar{j}} \left(\psi^i_z \partial_{\bar{z}} \bar{\phi}^j + \partial_z \phi^i \psi^{\bar{j}}_{\bar{z}} \right) \tag{6}$$

$$L' = L_A - 2t g_{i\bar{j}} \left(\partial_z \phi^i \partial_{\bar{z}} \bar{\phi}^j - \partial_{\bar{z}} \phi^i \partial_z \bar{\phi}^j \right) \tag{7}$$

$$S_A - S' = 2t \int_\Sigma d^2 z g_{i\bar{j}} \left(\partial_z \phi^i \partial_{\bar{z}} \bar{\phi}^j - \partial_{\bar{z}} \phi^i \partial_z \bar{\phi}^j \right) = t \int_\Sigma \phi^*(\omega) = t \int_{\phi(\Sigma)} \omega = t\omega \cdot \beta \tag{8}$$

For a fixed homology class β of $\phi(\Sigma)$, this term will simply contribute a prefactor $exp(-t\omega \otimes \beta)$. This number does not depend on the metric on the worldsheet, and the rest of the quantum measure, $e^{S'}$, is Q_A-exact. Due to the Q-exactness of the action, the semi-classical computation is exact and one can take the limit $t \to \infty$ to extract the results. The canonical commutation relations:

$$\left[\frac{dx^i(\sigma)}{d\tau}, x^j(\sigma') \right] = -\frac{i}{t} g^{ij} \delta(\sigma - \sigma') \tag{9}$$

$$\{\psi_\tau(\sigma), \chi(\sigma')\} = \frac{1}{t} \delta(\sigma - \sigma') \tag{10}$$

The Hilbert space is made up out of functionals $\Psi[x(\sigma), \dots]$, where x is a map from the interval as we have just described, and the $\cdots$ refer to the Grassmann fields (which play here the role of ghost fields). The Hamiltonian is obtained, as usual:

$$H = \int_0^\pi d\sigma T_{00} = \int_0^\pi d\sigma \left(-\frac{1}{t} g^{ij} \frac{\delta^2}{\delta x^i(\sigma) \delta x^j(\sigma)} + t g_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} \right) \tag{11}$$

We then see that string functionals with $\frac{dx^i}{d\tau} = 0$ cannot contribute: as $t \to \infty$, they get infinitely massive and decouple from the spectrum. Therefore, the map $x : I \to T^*M$ has to be constant and in particular it must be a point in M. A similar analysis holds for the Grassmann fields as well, and the conclusion is that the *string functionals are functions of the commuting and the anti-commuting zero modes*. Denoting them by q_a, χ^a, the string functional reduces to

$$\Psi = A^{(0)}(q) + \sum_{p=1}^{3} \chi^{a_1} \dots \chi^{a_p} A^{(p)}_{a_1 \dots a_p} \tag{12}$$

These functionals can be interpreted as differential forms on M. A differential form of degree p will have ghost number p. If we have N D-branes wrapping M, the above differential forms take values in the adjoint representation of the gauge group (i.e. they are valued in the $U(N)$ Lie algebra). On these functionals, the Q symmetry acts as the exterior differential, and $\{Q, \Psi\} = 0$ if the differential forms are closed. Of course in string field theory we do not restrict ourselves to functionals in the Q cohomology. We rather compute the string field action for arbitrary functionals, and then the condition of being in the Q-cohomology arises as a linearized equation of motion.

We are now ready to write the string field action for topological open strings on T^*M with Lagrangian boundary conditions specified by M. We have seen that the relevant string functionals are of the form (12). Since in string field theory the string field has ghost number one, we see that

$$\Psi = \chi^a A_a(q) \tag{13}$$

where $A_a(q)$ is a Hermitian matrix. In other words, the string field is just a $U(N)$ gauge connection on M. Since the string field only depends on commuting and anti-commuting zero modes, the integration of string functionals becomes ordinary integration of forms on M, and the star product becomes the usual wedge products of forms. We then have the following dictionary:

$$\Psi \to A, \quad Q_{BRST} \to d, \quad \star \to \wedge, \quad \int \to \int_M \tag{14}$$

The string field action is then the usual Chern–Simons action for A, and we have the following relation between the string coupling constant and the Chern–Simons coupling:

$$S_{CS} = \frac{1}{g_s} \int \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \qquad g_s = \frac{2\pi}{k+N} \tag{15}$$

This result is certainly remarkable. In the usual open bosonic string, the string field involves an infinite tower of string excitations. For the open topological string, the topological character of the model implies that all excitations decouple, except for the lowest-lying one. In other words, the usual reduction to a finite number of degrees of freedom that occurs in topological theories downsizes the string field to a single excitation. In physical terms, what is happening is that string theory reduces in this context to its pointlike limit, since the only relevant degree of freedom of the string is its zero mode, which describes the motion of a pointlike particle. The string field theory becomes a quantum field theory involving a finite number of fields.

However, since open topological string theory is a theory that describes open string instantons with Lagrangian boundary conditions, we should expect to have corrections to the above result due to nontrivial worldsheet instantons. It is easy to see that instantons $x : \Sigma \to T^*M$ such that $x(\partial\Sigma) \subset M$ are necessarily constant. Therefore, there are no instanton corrections to the Chern-Simons action that we derived above.

3 Conclusion

Topological string theories are in many ways similar to an ordinary string theories, one natural question which arises is: *are there also open topological strings which can end on D-branes?* To answer the above question rigorously, we would have to study boundary conditions on worldsheets with boundaries which preserve the Q-symmetry. In the A-model, one can only construct three-dimensional D-branes wrapping so-called "Lagrangean" submanifolds of M. Here, "Lagrangean" simply means that the Kahler form vanishes on this submanifold. Just like in ordinary string theory, when we consider open topological strings ending on a D-brane, there should be a field theory on the brane worldvolume describing the low-energy physics of the open strings. Moreover, since we are studying topological theories, one may expect such a theory to inherit the property that it only depends on a restricted amount of data of the manifolds involved.

In ordinary string theory, the worldvolume theory on N D-branes has a $U(N)$ gauge symmetry, so putting the ingredients together we can make the guess that the worldvolume theory is a three-dimensional, topological field theory with $U(N)$ gauge symmetry. There is really only one candidate for such a theory: the Chern-Simons theory. Recall that it consists of a single $U(N)$ gauge field.

We, following Witten, showed that this is indeed the theory one obtains. In fact, *this theory actually describes the full topological string field theory on the D-branes, even without going to a low-energy limit.*

After the analysis, we know that Chern-Simons theory on S^3 is a topological open string theory on T^*S^3. Gopakumar and Vafa conjectured [11] that Chern–Simons theory on S^3 is equivalent to closed topological string theory on the resolved conifold. The conjecture can be justified physically by doing a careful analysis of the worldsheet description of the topological strings involved in the duality [12]. Putting together the cut-and-paste approach to toric Calabi– Yau manifolds with the large-N duality relating Chern–Simons theory and topological strings, give us a building block for topological string amplitudes on those geometries. This building block is called the topological vertex [13]. Future application are described in [14, 15, 16, 17, 18, 19].

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