Hamiltonian reduction and solution for open Toda chain

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1 Introduction

Open Toda chain is an integrable many-body system with exponential interaction. Hamiltonian of the system is the following:

$$H^{\mathrm{T}} = \sum_{i=1}^{n} \frac{p_i^2}{2} + \sum_{i=1}^{n-1} g_i^2 e^{2(q_i - q_{i+1})} \tag{1.1}$$

with the symplectic form:

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i. \tag{1.2}$$

It gives rise to the standard equations of motions:

$$\dot{q}_i = \frac{\partial H^T}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H^T}{\partial q_i}. \tag{1.3}$$

Our aim is to solve the equations of motions (1.3) for open Toda chain using hamiltonian reduction. For more information about the hamiltonian reduction and the moment map see [5].

2 Two-particle system

We start with the elementary example of two particle Toda chain. Our main aim in this section will be showing how to obtain Toda chain hamiltonian geometrically via considering free motion in the space of higher dimension(geodesic flows).
Hamiltonian of two particle Toda chain is the following:

$$H = \frac{1}{2}(p_1^2 + p_2^2) + g_1^2 e^{2(q_1 - q_2)}, \tag{2.1}$$

the equations of motions are:

$$\dot{q}_i = p_i, \quad \dot{p}_1 = -2g_1^2 e^{2(q_1 - q_2)}, \quad \dot{p}_2 = 2g_1^2 e^{2(q_1 - q_2)}. \tag{2.2}$$

After introducing a new variable $q = q_2 - q_1 + c$, $c = const$ we get the hamiltonian:

$$H = \tfrac{1}{2}p^2 + g^2 e^{-2q}, \quad p = \dot{q}. \tag{2.3}$$

Let us show how to obtain this hamiltonian geometrically. Consider a free motion on the upper bowl of a two-cavity hyperboloid $\{x_0^2 - x_1^2 - x_2^2 = 1, x_0 > 0\}$. The kinetic energy is determined by the metric on the hyperboloid and is equal to $T = -\frac{1}{2}(\dot{x}_0^2 - \dot{x}_1^2 - \dot{x}_2^2)$. Intriduce the orispecrical coordinates:

$$x_0 = \cosh q + \tfrac{z^2}{2} e^q, \quad x_1 = \sinh q - \tfrac{z^2}{2} e^q, \quad x_2 = e^q z, \tag{2.4}$$

in this coordinates the kinetic energy reads as:

$$T = \frac{1}{2}(\dot{q}^2 + \mu e^{-2q}), \tag{2.5}$$

where $\mu = \dot{z} e^{2q} = const$. Finally, we obtained the hamiltonian for two particle Toda chain.
In next sections we will explain how to get the hamiltonian of n particle Toda chain using free motion totally analogical to the two particle case.

3 Dynamics are on the extended space

To obtain the many-body open Toda chain, similarly to the section 2, we consider the extended space X - symmetrical, positive-definite matrices with the determinant equal to 1. The group $G = SL(n, \mathbb{R})$ acts on the space X transitively $x \to g x g^T$, where $x \in X \quad g \in SL(n, \mathbb{R}) = G$. Let Z - subgroup of uppertriangular matrices with "1" on the diagonal and H - subgroup of diagonal matrices in G. Every element of X may be represented in the following form(Iwasava decomposition):

$$x = z(x) h^2(x) z^T(x), \quad h \in H, z \in Z. \tag{3.1}$$

In this way h and z are the coordinates in the space X, this system of coordinates is called orispherical. Note that groups $SL(2, \mathbb{R})$ and $SO(2,1)$ are locally isomorphic and systems of coordinates decribed in sections 2 and 3 are equivalent. Coordinate $h(x)$ is called the orispherical projection of the element x. Instead of $h(x)$ it is more convinient to consider $h(x) = e^{q(x)}$. For the orispherical projection we have the following formula:

$$h_j(x) = (\frac{\Delta_{n-j+1}}{\Delta_{n-j}})^{\frac{1}{2}}, \tag{3.2}$$

where $h(x) = \mathrm{diag(h_1(x), .., h_n(x))}$ and Δ_i - lower corner minor of the order i of the matrix x, $\Delta_0 = 1$, the proof of this statement can be find, for example, in [3].
Consider the cotangent bundle T^*X, it's elements may be considered as pairs (x, y), $x \in X \quad y \in T_x^* X$ and one-form on it $\theta = -\frac{1}{2}\mathrm{tr}(y dx^{-1})$ and the symplectic form on T^*X: $\Omega = d\theta = -\frac{1}{2}\mathrm{tr}(dy \wedge dx^{-1})$. It is evident that these forms are equivalent under the following transformations:

$$\begin{aligned} x &\to g x g^T, \\ y &\to g y g^T. \end{aligned} \tag{3.3}$$

Consider the Hamiltonian

$$H = \frac{1}{2}\mathrm{tr}(y x^{-1} y x^{-1}). \tag{3.4}$$

Together with the symplectic form desribed in the above the hamiltonian defines the dynamic:

$$\begin{aligned} \dot{x} &= 2y, \\ \dot{y} &= 2y x^- 1 y, \end{aligned} \tag{3.5}$$

from this equations one can deduce $\frac{d}{dt}\dot{x}x^-1 = 0$, then it is possible to find x:

$$x(t) = b\exp(2at)b^T, \tag{3.6}$$

where $b \in G$, $a \in T_eX$. From the form of θ one can find that $\mathrm{tr}(yx^{-1})^k$ with $k > 1$ are the integrals of motion.

4 Hamiltonian reduction to the Toda system

Action of the group G on T^*X is as follows:

$$x \to gxg^T, \quad y \to gyg^T. \tag{4.1}$$

The action gives rise to the flow:

$$(\dot{x})^{-1} = \mathbf{g}^T x^{-1} - x^{-1}\mathbf{g}, \quad \dot{y} = \mathbf{g}y + y\mathbf{g}^T, \tag{4.2}$$

where $\mathbf{g}$ lies in the Lie algebra Lie(G). Using the symplectic potential *theta* one can obtain that this flow is generated by the hamiltonian:

$$F(x,y,\mathbf{g}) = -\frac{1}{2}\left[\mathrm{tr}(x^{-1}y\mathbf{g}^T) + \mathrm{tr}(yx^{-1}\mathbf{g})\right]. \tag{4.3}$$

Let $G = SL(n,\mathbb{R})$, Z - upper triangular matrices with "1" on diagonals and Z' - lower triangular matrices with "1" on diagonals. Let $\mathcal{Z}$ and $\mathcal{Z}'$ thier Lie algebras respectively. Define the moment map by the following formula:

$$F(x,a) =< \Psi(x), a >, \tag{4.4}$$

$x \in M$, a-belongs to the Lie algebra Lie(G), then

$$\Psi_g : (x,y) \to \mu + \mu', \tag{4.5}$$

where $\mu = (x^{-1}y)_+ \quad \mu' = (yx^{-1})_-$. Signs $+$ and $-$ mean the projection onto the upper and lower triangular matrices respectively.

$$\mu_{jk} = g_j\delta_{j,k-1}. \tag{4.6}$$

Let us use the hamiltonian reduction with respect to the group Z.
Proposition 1: Submanifold $\Psi^{-1}(\mu)$ in T^*X for μ and Ψ from the above are described by the matrices:

$$x = ze^{2q}z^T, \quad y = z\tilde{y}z^T, \tag{4.7}$$

where $q = \mathrm{diag(q_1,..,q_n)}$, $\tilde{y}_{kj} = p_k e^{2q_k}\delta_{kj} + g_j e^{2q_j}\delta_{k,j+1} + g_{j-1}e^{2q_{j-1}}\delta_{k+1,j}$, $(p_1,..,p_n) \in \mathbb{R}^n$.
Proposition 2: Reduced phase space $\tilde{M} = \Psi^{-1}(\mu)/Z$ is parametrized by two vectors $q = (q_1,..,q_n)$ and $p = (p_1,..,p_n)$ with $\sum\limits_j q_j = \sum\limits_j p_j = 0$ or by two matrcies $\exp(2q)$ and $\tilde{y}$.
Proposition 3:With the moment map Ψ hamiltonian $H = \frac{1}{2}\mathrm{tr}(yx^{-1}yx^{-1})$ becomes the hamiltonian of the Toda chain, the form Ω becomes the canonical form $\tilde{\Omega} = dp \wedge dq$ on $\tilde{M}$.
Proofs of all the propositions can be find in the [1]. Let p^0 and q^0 -the initial data for the Toda chain and Jacobi matrix:

$$b_{jk} = p_j^0\delta_{jk} + g_{j-1}e^{q^0_{j-1}-q^0_j}\delta_{j,k+1} + g_j e^{q^0_j - q^0_{j-1}}\delta_{j,k-1}. \tag{4.8}$$

Then the following theorem holds:
Theorem 1: Let $\Delta_j(t)$ - lower corner minors of $\exp(2bt)$, then the solution of the Toda chain is

$$q_k(t) = q_k(0) + \frac{1}{2}\ln\frac{\Delta_{n-k+1}}{\Delta_{n-k}(t)}, \Delta_0 = 1. \tag{4.9}$$

Proof of the theorem can also be find in the [1].

5 Conclusion

Using the hamiltonian reduction formalism we desribed how to obtain the open Toda chain from the geodesic dynamics on the extended homogeneous space X - symmetrical, positive-definite matrices with the determinant equal to 1. The hamiltonian reduction ables one to solve the Toda chain explicitely and write and solution in the form of (4.9). It is worth mentioning that we considered the "usual" open Toda chain, however one can write the generalized open Toda chain hamiltonian with use of root systems of the complex simple Lie algebras. The Toda system described above is related to the root system A_n of the Lie algebra $sl(n+1)$. The generalized Toda chains can obtained in the same by the means of the hamiltonian reduction, the exact solutions can also be find, for more information about generalized Toda chains and the hamiltonian reduction see [2]. Also, let us mention that hamiltonian reduction can be used to obtain not only Toda chain systems, but also integrable many-body systems of Calogero type [4].

References

[1] M.A. Olshanetsky, A.M. Perelomov, Toda chain as a reduced system, Theoret. and Math. Phys., 45:1 (1980), 843854

[2] M.A. Olshanetsky, A.M. Perelomov, Explicit Solutions of Classical Generalized Toda Models,lnventiones math. 54, 261-269 (1979)

[3] I. M. Gel'fand, M. A. Naimark, Unitary representations of the classical groups, Trudy Mat. Inst. Steklov., 36, Acad. Sci. USSR, MoscowLeningrad, 1950, 3288

[4] D. Kazhdan, B. Kostant, S. Sternberg., Hamiltonian group actions and dynamical systems of calogero type, Commun. Pure. Appl. Math., 31, 481, 1978.

[5] V.I. Arnold, Mathematical Methods of Classical Mechanics, Science, 1974