Plan
Nakajima quiver varieties: definition and examples

1 Hilbert scheme of points on the affine plane

Resolution of singularities and Hilbert schemes of points

- We have natural object $S_n(X) = X^{\times n}/S_n$ — natural configuration space of n points. We will usually consider the simplest case $X = \mathbb{A}^k$.
- Case $\dim X = 1$: no singularities, coordinates — elementary symmetric polynomials (on the map) $e_1, \dots e_n$.
- Case $\dim X = 2$ — singularities when points coincide: conic singularity via $\mathbb{C}/S_2$.
- Resolution: Hilbert schemes
$$X^{[n]} = (\mathbb{A}^k)^{[n]} = \{I \subset R = \mathbb{C}[z_1 \dots z_k] | \dim R/I = n\}. \tag{1}$$
— 0-dimensional subsheme of length n — generally n points. In case $k = 1$ it is $S_n(\mathbb{A}^{\times n})$. It is always non-singular for non-singular X — see below.
- Example: take ideal $z_2 + Pol_{n-1}(z_1) = 0$ and $z_1^n + a_{n-1}z_1^{n-1} + \dots = 0$. This corresponds to the n points, generally distinct. Take $a_i = 0$, then $n = 2$ and then shift $z_2 = 0$. This is coinciding point. This is ideal $J = (z_1^2; z_2 - \alpha z_1) = \{f \in \mathcal{O}_X | f(0) = 0, df_0(v) = 0\}$ of colength 2 — blow-up. Last equality — from the local data: (y_1, y_2^2) in the turned basis.
- Hilbert-Chow morphism π
$$\pi : X^{[n]} \mapsto S^n X \tag{2}$$
given by
$$\pi(Z) = \sum_{x \in X} \text{length}(Z_x)[x] \tag{3}$$
where length(Z_x)=colength of ideal Z, restricted on the vicinity of x, in $\mathcal{O}_X$.

Hilbert scheme of points on the plane: handy description

- **Theorem 1.1** $(\mathbb{A}^2)^{[n]} = H$ *where*
$$H = \{(B_1, B_2, i) | [B_1, B_2] = 0; \textbf{\textit{stability}}: \textit{no proper subspace } S \textit{ s.t.} B_{1,2}(S) \subset S, \operatorname{im} i \subset S\}/GL_n(\mathbb{C}), \tag{4}$$
where $B_1, B_2 \in \text{End}(\mathbb{C}^n), i \in \text{Hom}(\mathbb{C}, \mathbb{C}^n)$ *and action of* $GL_n(\mathbb{C})$ *is* $g \cdot (B_1, B_2, i) = (gB_1g^{-1}, gB_2g^{-1}, gi)$

- Correspondence at the level of sets: Point in $(\mathbb{A}^2)^{[n]}$ — ideal I. $V = \mathbb{C}[z_1, z_2]/I$ — n-dimensional space. Multiplication by $z_{1,2} \bmod I$ give us commuting $B_{1,2}$. $i(1) = 1 \bmod I$ and stability condition also holds (because mult 1 on z_1 and z_2 spanning all $\mathbb{C}[z_1, z_2]$). Identification of vector spaces V_I arised from different ideals is not necessary because we are interested in $GL_n(\mathbb{C})$-orbits.

  Conversely: From (B_1, B_2, i) define $\phi : \mathbb{C}[z_1, z_2] \mapsto \mathbb{C}^n : \phi(f) = f(B_1, B_2)i(1)$. By stab. cond. $\operatorname{im}\phi = \mathbb{C}^n$. $I = \ker\phi$ so $\dim \mathbb{C}[z_1, z_2]/I = n$. By the stability condition
$$I = \{f(z) \in \mathbb{C}[z_1, z_2] | f(B_1, B_2) = 0\} \tag{5}$$

- Non-singularity of H: Map $B_1, B_2 \mapsto [B_1, B_2]$ has constant rank. Calculating cokernel of differential
$$\{\xi | \operatorname{tr}(\xi([B_1, \delta B_2] + [\delta B_1, B_2])) = 0\} \Leftrightarrow \{\xi | [\xi, B_1] = [\xi, B_2] = 0\} \tag{6}$$
  Assign $\xi \mapsto \xi(i(1))$ ($i(1) \neq 0$ by stab.). Inverse of this map is given by
$$\xi(B_1^l B_2^m i(1)) = B_1^l B_2^m v, l, m \in \mathbb{Z}_{\geq 0} \tag{7}$$
  so coker has dim $= n$ and variety without factorization is non-singular. Stabilizer of $GL_n(\mathbb{C})$ action is trivial. $gB_{1,2}g^{-1} = B_{1,2}, gi = i$ so $\operatorname{im} i \subset \ker(g - id)$ and the ker is inv under $B_{1,2}$, so from stab. $g = id$.

- Tangent space to $(\mathbb{A}^2)^{[n]}$ in point (B_1, B_2, i). Complex:
$$\operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^n) \mapsto_{d_1} \operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^n) \oplus \operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^n) \oplus \mathbb{C}^n \mapsto_{d_2} \operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^n) \tag{8}$$
  with $d_1(\xi) = ([\xi, B_1], [\xi, B_2], \xi i)$ — derivative of GL_n action, $d_2(C_1, C_2, I) = [B_1, C_2] + [C_1, B_2]$ — derivative of commutator map. $Ker d_1 = 0$, tangent space$=$ $\operatorname{Ker}(d_2)/\operatorname{Im}(d_1)$ with dimension $2n = 2n^2 + n - (n^2 - n) - n^2$.

- Example: $n = 2$: distinct points stratum. Either B_1 or B_2 have distinct eigenvalues. Then $B_1 = diag(\lambda_1, \lambda_2)$, $B_2 = diag(\mu_1, \mu_2)$, $i(1) = (1, 1)^t$.
$$I = \{f(z) \in \mathbb{C}[z_1, z_2] | f(\lambda_1, \mu_1) = 0, f(\lambda_2, \mu_2) = 0\} = (z_1(z_1 - \lambda), \lambda z_2 - \mu z_1), \tag{9}$$
  where we shifted ($\lambda_1 = \mu_1 = 0$) in the last equality.

- Example: $n = 2$: coinciding points $B_1 = \lambda E, B_2 = \mu E$ violates the stability condition. So we have that
$$B_1 = \begin{pmatrix} \lambda & \alpha \\ 0 & \lambda \end{pmatrix}, \quad B_1 = \begin{pmatrix} \mu & \beta \\ 0 & \mu \end{pmatrix}, \tag{10}$$
  for $(\alpha, \beta) \in \mathbb{C}^2 \backslash \{0, 0\}$ and $i = (0, 1)^t$. Remaining factor by $diag(a, 0)$ and after factorization we obtain $(\alpha : \beta) \in \mathbb{CP}^1$
$$I = \{(z_1 - \lambda)^2, (z_2 - \mu)^2, \beta(z_1 - \lambda) - \alpha(z_2 - \mu)\} = \{f | f = (\alpha : \beta)\nabla f|_{\lambda,\mu} = 0\} \tag{11}$$

- Hilbert-Chow morphism. Upper-triangular form of the commutative matrices $[B_1, B_2]$ give us eigenvalues λ_i, μ_i. Morphism is given by (λ_i, μ_i). All points are distinct — matrices semisimple — we have isomorphism.

- $n-1$ dim component of $\pi^{-1}(n[0])$. It corresponds to $B_1 = E^{(1)}, B_2 = \sum_{i=1}^{n-1} a_i E^{(i)}$ and $i = (0 \dots 0, 1)^t$. As an ideal it is given by $\mathcal{J} = (z_1^n, z_2 - \sum_{i=1}^{n-1} a_i z_1^i)$.

Giseker variety: definition and example

- Gieseker variety (or instanton moduli space variety): the simplest case of Nakajima quiver variety

  Definition 1.1 *$(\mathbb{A}^2)^{[n]} = H$ where*

  $$\mathcal{M}(r,n) = \{(B_1, B_2, i, j) | [B_1, B_2] + ij = 0; \textbf{\textit{stability}}: \textit{no proper subspace } S \textit{ s.t.} B_{1,2}(S) \subset S, \operatorname{im} i \subset S\}/G \tag{12}$$

  where $B_1, B_2 \in \operatorname{End}(\mathbb{C}^n), i \in \operatorname{Hom}(\mathbb{C}^r, \mathbb{C}^n), j \in \operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^r)$ and action of $GL_n(\mathbb{C})$ is $g \cdot (B_1, B_2, i, j) = (gB_1g^{-1}, gB_2g^{-1}, gi, jg^{-1})$

  Figure of quiver

- Equivalence with previous definition: $\mathcal{M}(1,n) = (\mathbb{C}^2)^{[n]}$. That is because

  Lemma 1.1 *Let $r = 1$ and S is generated from $i(1)$ by $B_{1,2}$. Then $j|_S = 0$.*

  $j\hat{B}i = 0$ by induction from $ji = tr(ij) = 0$ where $r = 1$ is important.

- Description of $S_n(\mathbb{C}^2)$

  $$S_n(\mathbb{C}^2) = \{(B_1, B_2, i, j) | [B_1, B_2] + ij = 0\} // GL_n(\mathbb{C}), \tag{13}$$

  where $//$ — set of closed $GL_n(\mathbb{C})$ orbits.

  Proof: Take a close orbit and S, then $\mathbb{C}^n = S \oplus S^\perp$

  $$B_{1,2} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, \quad i = (*, 0)^t, \quad j = (0, *). \tag{14}$$

  Take $g(t) = diag(1, t)$ and $t \mapsto 0$ and then $g(t') = t'^{-1}$ and $t' \mapsto 0$, then we obtain in this orbit $i = j = 0$ with $[B_1, B_2] = 0$ and we can make them both upper-triangular. Taking $g(t) = diag(1, t, t^2 \dots)$ and $t \mapsto 0$ we make $B_{1,2}$ semisimple.

2 Generalization: Quiver variety

From Gieseker to quiver varieties Now we generalize the previous example. Instead of self-loop graph we take quiver Q and introduce space M.

Graph Q (edges and orientation H, vertices $1 \dots n$) with orientation Ω, $\mathbf{v}$ — dim of tuple V_k — hermitian vector spaces.

$$M = \left(\bigoplus_{h \in H} \operatorname{Hom}(V_{out(h)}, V_{in(h)}) \right) \oplus \left(\bigoplus_{k=1}^{n} \operatorname{Hom}(V_k, W_k) \oplus \operatorname{Hom}(W_k, V_k) \right) = (B_h, i_k, j_k). \tag{15}$$

$\dim M = \langle \mathbf{v}, A\mathbf{v} \rangle + 2\langle \mathbf{v}, \mathbf{w} \rangle$, $C = 2I - A$. Symplectic structure on M

$$\omega((B,i,j),(B',i',j')) = \sum_{h \in \mathbb{H}} \operatorname{tr}(\epsilon(h) B_h B'_{\overline{h}}) + \sum_{k=1}^{n} \operatorname{tr}(i_k j'_k - i'_k j_k), \tag{16}$$

so that $M = M_\Omega \oplus M_{\overline{\Omega}}$ — natural Lagrangian subspaces.

Group $G_\mathbf{v} = \prod \operatorname{GL}(V_k)$ acts by

$$(B_h, i_k, j_k) = (g_{in(h)} B_h g^{-1}_{out(h)}, g_k i_k, j_k g_k^{-1}) \tag{17}$$

Introduce moment map

$$\mu(B,i,j) = \sum_{h \in H, k = in(h)} \epsilon(h) B_h B_{\overline{h}} + i_k j_k \tag{18}$$

Then analog of $S_n(\mathbb{C}^2)$ is $\mathfrak{M}_0$ defined by

$$\mathfrak{M}_0(\mathbf{v}, \mathbf{w}) = \mu^{-1}(0) // G_\mathbf{v} \tag{19}$$

Trivial line G-bundle $\mu^{-1}(0) \times \mathbb{C}$ with $g(B,i,j,z) = (g(B,i,j), z \prod_{k=1} \det g_k)$. Stable points

$$\mu^{-1}(0)^s = \{(B,i,j) \in \mu^{-1}(0) | \overline{G(B,i,j,z)} \cap (\mu^{-1}(0) \times \{0\}) = \varnothing \text{for} z \neq 0\} \tag{20}$$

Definition 2.1 *Nakajima quiver variety is given by*

$$\mathfrak{M}(\mathbf{v}, \mathbf{w}) = \mu^{-1}(0)^s / G_\mathbf{v} \tag{21}$$

For stable points G-stabilizer is trivial and $d\mu$ is surjective so $\dim \mu^{-1}(0)^s = \langle v, 2w + (I - C)v \rangle$ and $\dim \mathfrak{M} = \langle v, 2w - Cv \rangle$

Projective morphism $\mathfrak{M} \mapsto \mathfrak{M}_0$ exists

Quiver varieties of type A_n, e.g. partial flag variety. Dynkin quiver $1 \leftarrow 2 \leftarrow \dots \leftarrow n$, then $\mathfrak{M}_0 = pt$. Take $\mathbf{v} = (v_1, v_2 \dots v_n)$ and $\mathbf{w} = (r, 0, \dots 0)$, $r > v_1 > \dots v_n > 0$. Framed representation of quiver $(x_{1,2}, \dots x_{n-1,n}, j)$. Take collection of the vector spaces

$$F_i = \operatorname{Im}(j x_{1,2} \dots x_{i-1,i}) \subset \mathbb{C}^r. \tag{22}$$

form partial flag $F_1 \subset F_2 \subset \dots F_n$ Stability condition: injectivity of the maps, so $\dim F_i = v_i$. Introduce flag variety $\mathcal{F}_\mathbf{v}(n, W)$. We obtain that $\mathcal{R}(\mathbf{v}, \mathbf{w}) = \mathcal{F}_\mathbf{v}(n, W)$.